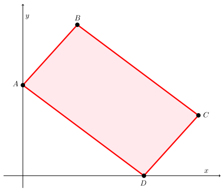


Problem of the Week
Problem E and Solution
Positioned Differently

Problem

Often we draw parallelograms so that two of the sides are either horizontal or vertical. The parallelogram, $ABCD$, is positioned differently. A lies on the positive y-axis, D is on the positive x-axis, and B and C lie in the first quadrant. Three of its vertices, A, B, and D are located at $(0, 30)$, $(k, 50)$ and $(40, 0)$, respectively. The area of $ABCD$ is 1340 units^2. If $k > 0$, determine the coordinates of B and C.

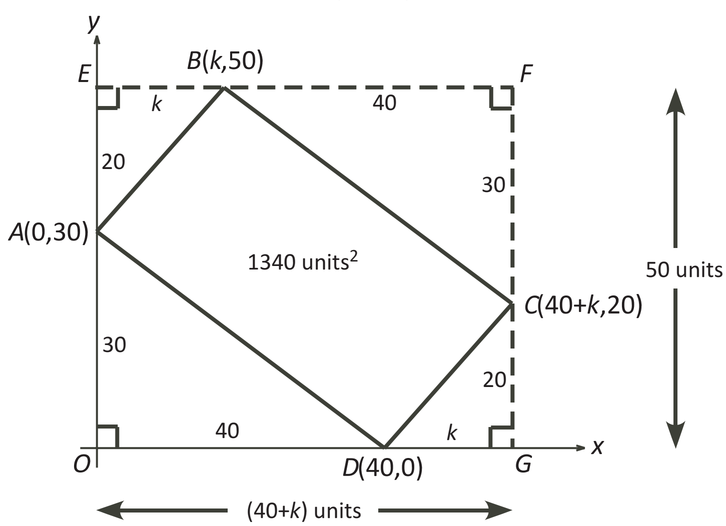
Solution

Since $ABCD$ is a parallelogram, $AB = DC$ and $AB \parallel DC$. We can use this to find the coordinates of C. To get from A to B, we go up 20 units and right k units. Therefore, to get from D to C we do the same. C is located at $(40 + k, 20)$.

In the solution, we will use a method known commonly as "*completing the rectangle*".

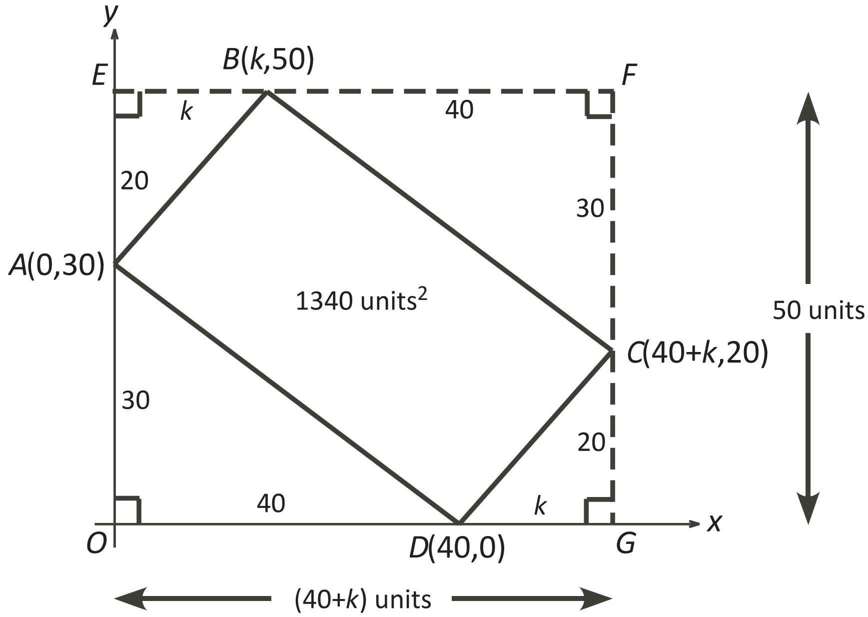
Enclose $ABCD$ in rectangle $OEFG$ such that OE is on the positive y-axis passing through A, EF is parallel to the positive x-axis passing through B, FG is parallel to the positive y-axis passing through C, and OG lies along the positive x-axis passing through D.

This information is presented on the following diagram.



The y coordinate of B is the distance from the x-axis to EF and also the height, GF, of rectangle $OEFG$. It follows that $GF = 50$ units. Similarly, the x coordinate of C is the distance from the y-axis to GF and also the width, OG, of rectangle $OEFG$. It follows that $OG = (40 + k)$ units. The other dimensions follow. (This information is already marked on the above diagram.)

The diagram from the first page is repeated here.



We can now put the information together using areas to determine the value of k.

$$\begin{aligned}
\text{Area } OEFG &= \text{Area } \triangle AEB + \text{Area } \triangle BFC + \text{Area } \triangle CGD + \text{Area } \triangle DOA + \text{Area } ABCD \\
FG \times OG &= \frac{AE \times EB}{2} + \frac{BF \times FC}{2} + \frac{CG \times GD}{2} + \frac{DO \times OA}{2} + 1340 \\
50 \times (40 + k) &= \frac{20 \times k}{2} + \frac{40 \times 30}{2} + \frac{20 \times k}{2} + \frac{40 \times 30}{2} + 1340 \\
2000 + 50k &= 10k + 600 + 10k + 600 + 1340 \\
2000 + 50k &= 20k + 2540 \\
30k &= 540 \\
k &= 18
\end{aligned}$$

Therefore, the value of k is 18 and coordinates of B and C are $B(18, 50)$ and $C(58, 20)$, respectively.

The solver may have approached the problem using linear equations and intersections. This is a very acceptable solution to the problem. However, in this problem, that approach probably would involve considerably more work.