# Problem of the Week
## Problem E and Solution
## Search and Swap

### Problem

Randi has a deck consisting of 10 cards. One side of each card is red and the other side of each card has one of the letters A, B, C, D, E, F, G, H, I, or J on it. Each letter occurs exactly once. The cards are shuffled and placed letter-side down on a table from left to right.

Every time Randi looks for a letter, she turns over cards one by one starting with the leftmost card and moving to the right. If a card does not have the letter she is looking for on it, Randi puts it back letter-side down in the same location and continues with the next card. If a card does have the letter she is looking for on it, Randi swaps the locations of this card and the card on its immediate left placing both cards letter-side down. One exception is when she finds the letter she is looking for on the leftmost card. In this case, Randi puts the card back letter-side down in the same location and no swap occurs. Either way, once Randi finds the letter she is looking for, she does not look at any more cards. Also, Randi never remembers the locations of any cards on the table.

If the ten cards begin in some unknown order and Randi searches for each of the ten letters exactly once, what is the maximum possible number of cards that Randi looks at?

### Solution

If no swaps were required as a result of finding a card, how many cards would Randi have to look at in total?

At some point she is looking for the card in position 1. She would have to look at 1 card to find it. At some point she is looking for the card in position 2. She would have to look at 2 cards to find it. At some point she is looking for the card in position 3. She would have to look at 3 cards to find it. This continues until at some point she is looking for the card in position 10. She would have to look at 10 cards to find it. To locate all 10 cards, Randi would have to look at $1 + 2 + 3 + \cdots + 10 = 55$ cards.

Since Randi only looks for each letter exactly once, swapping the position of one letter with the position of another letter can only have the effect of increasing the number of cards looked at for the letter on the preceding card by one. The number of cards looked at to find other letters would not be affected. Therefore, swapping can only increase the number of cards looked at (by one) for all but the first search. These means swapping can increase the number of cards looked at by at most 9 in total making the maximum total number of cards looked at equal to $55 + 9 = 64$.

On the next page, an illustration of how this maximum can be achieved is illustrated.

Is 64 an achievable maximum?

Put the cards in order, left to right, from A to J.

A B C D E F G H I J

Now search for each letter in order from B to J and search for A last.

Since B is in the second position, we must look at 2 cards to find it. We then swap A and B.

B A C D E F G H I J

Since C is in the third position, we must look at 3 cards to find it. We then swap A and C.

B C A D E F G H I J

Since D is in the fourth position, we must look at 4 cards to find it. We then swap A and D.

B C D A E F G H I J

Since E is in the fifth position, we must look at 5 cards to find it. We then swap A and E.

B C D E A F G H I J

Since F is in the sixth position, we must look at 6 cards to find it. We then swap A and F.

B C D E F A G H I J

Since G is in the seventh position, we must look at 7 cards to find it. We then swap A and G.

B C D E F G A H I J

Since H is in the eighth position, we must look at 8 cards to find it. We then swap A and H.

B C D E F G H A I J

Since I is in the ninth position, we must look at 9 cards to find it. We then swap A and I.

B C D E F G H I A J

Since J is in the tenth position, we must look at 10 cards to find it. We then swap A and J.

B C D E F G H I J A

Finally, since A is in the tenth position, we must look at 10 cards to find it. We then swap A and J (again).

B C D E F G H I A J

We have looked at a total of $2+3+4+5+6+7+8+9+10+10 = 64$ cards to locate each of the cards.

On the next page, we look at a possible extension and a connection to Computer Science.

**Extension**:

Suppose you have $n$ cards, each with something different on them. You lay the cards out in a similar manner to how we handled the 10 different cards. You search for each of the different cards, one at a time. What is the maximum number of cards you must look at in order to locate all of the cards using the search described in the problem?

**Connection to Computer Science**:

The problem first appeared on the Beaver Computer Challenge in 2012. You can see the problem and others like it at http://www.cemc.uwaterloo.ca/contests/bcc.html.

One of the fundamental problems in computer science is how to organize data in order to search within it quickly. There are many ways to do this: using binary trees, splay trees, skip lists, sorted arrays, etc. The technique outlined in this problem is the idea of moving found items closer to the "front," with the assumption that if we search for something once, it is quite likely that the same item will be searched for again. The transpose (swap) heuristic used by Randi in this problem is one technique for doing this. Other heuristics include move-to-front, which moves a found element to the very front of the list. Moreover, this problem highlights the process of performing worst-case analysis for an algorithm. Computer scientists care about "what is the worst possible input for this algorithm, and how long will it take to execute on that input?" In this question, we are asking about the worst-case performance of the transpose heuristic on a list of size 10.