Problem of the Week
Problem E and Solution
E Z Does It Again

Problem

E Z Dealer has a deck consisting of 100 cards numbered from 1 to 100. Each card has the same number printed on both sides. One side of the card is yellow and the other side of the card is red. E Z places all the cards, red side up, on the table. He first turns over every card that has a number which is a multiple of 2. He then examines all the cards, and turns over every card that has a number which is a multiple of 3. He again examines all the cards, and turns over every card that has a number which is a multiple of 4. Finally, he examines all the cards and turns over every card that has a number which is a multiple of 5. After E Z has finished, how many cards have the red side facing up?

Solution

If a card number is a multiple of 2, 3, 4 and 5, it will be flipped four times. This card will go from red to yellow to red to yellow to red again. So the card will still be red once E Z has finished.
If a card number is a multiple of exactly three of 2, 3, 4 and 5, it will be flipped three times. This card will go from red to yellow to red to yellow. So the card will be yellow once E Z has finished.
If a card number is a multiple of exactly two of 2, 3, 4 and 5, then it will be flipped twice. This card will go from red to yellow to red again. So the card will still be red once E Z has finished.
If a card number is a multiple of exactly one of 2, 3, 4 and 5, it will be flipped once. This card will go from red to yellow. So the card will be yellow once E Z has finished.
If a card number is a multiple of none of 2, 3, 4 and 5, then this card will not be flipped and so the card will still be red once E Z has finished.

To determine how many cards have the red side facing up once E Z has finished, let's determine how many cards have the yellow side facing up once E Z has finished. To do so, we need to determine how many card numbers are multiples of exactly three of 2, 3, 4 and 5 and how many cards are multiples of exactly one of 2, 3, 4 and 5.

Let's consider the cases:

- A card number is a multiple of 2, 3 and 4, but not 5
  If a card number is a multiple of 2, 3 and 4, then it must be a multiple of 12, the lowest common multiple of 2, 3 and 4. So we are want card numbers that are multiples of 12 but not 5. If a card number is a multiple of 12 and 5, then it is a multiple of $12 \times 5 = 60$. So we want all multiples of 12 that are not multiples of 60.
  There are 8 multiples of 12 from 1 to 100, but one is 60. So there are $8 - 1 = 7$ numbers that are multiples of 2, 3 and 4, but not 5.
- A card number is a multiple of 2, 3 and 5, but not 4
  If a card number is a multiple of 2, 3 and 5, then it must be a multiple of 30, the lowest common multiple of 2, 3 and 5. So we want all multiples of 30 that are not multiples of 4.
  There are 3 multiples of 30 from 1 to 100, but one is 60, which is also a multiple of 4. So there are 2 numbers from 1 to 100 that are multiples of 2, 3 and 5, but not 4.

- A card number is a multiple of 2, 4 and 5, but not 3
  If a card number is a multiple of 2, 4 and 5, then it must be a multiple of 20, the lowest common multiple of 2, 4 and 5. So we want all multiples of 20 that are not multiples of 3.
  There are 5 multiples of 20 from 1 to 100, but one is 60, which is a multiple of 3. So there are 4 numbers from 1 to 100 that are multiples of 2, 4 and 5, but not 3.
- A card number is a multiple of 3, 4 and 5, but not 2
  It is not possible for a card number to be a multiple of 4 but not 2. So there are no card numbers in this case.
- A card number is a multiple of 2 but not 3, 4, or 5
  There are 50 numbers from 1 to 100 which are multiples of 2 and 25 numbers from 1 to 100 which are multiples of 4 (and thus 2). So there are $50 - 25 = 25$ numbers from 1 to 100 multiples of by 2 but not 4. These are
  $$\{2, 6, 10, 14, 18, 22, 26, 30, 34, 38, 42, 46, 50, 54, 58, 62, 66, 70, 74, 78, 82, 86, 90, 94, 98\}.$$
  We need to remove numbers that are still multiples of 3 or 5. After doing so we are left with
  $$\{2, 14, 22, 26, 34, 38, 46, 58, 62, 74, 82, 86, 94, 98\}.$$
  So there are 14 numbers from 1 to 100 that are multiples of 2 but not 3, 4 or 5.
- A card number is a multiple of 3 but not 2, 4, or 5
  There are 33 multiples of 3 from 1 to 100,
  $$\{3,\ 6,\ 9,\ 12,\ 15,\ \cdots,87,\ 90,\ 93,\ 96,\ 99\}.$$
  In this group of multiples, there are 17 numbers that are odd.
  So there are 17 numbers from 1 to 100 that are multiples of 3 but not 2. These numbers are
  $$\{3, 9, 15, 21, 27, 33, 39, 45, 51, 57, 63, 69, 75, 81, 87, 93, 99\}.$$
  We still need to remove numbers that are multiples of 5. After doing so we are left with
  $$\{3, 9, 21, 27, 33, 39, 51, 57, 63, 69, 81, 87, 93, 99\}.$$
  So there are 14 numbers from 1 to 100 that are multiples of 3 but not 2, 4 or 5.
- A card number is a multiple of 4 but not 2, 3, or 5
  It is not possible for a card number to be a multiple of 4 but not 2. So there are no card numbers in this case.
- A card number is a multiple of 5 but not 2, 3, or 4
  There are 20 multiples of 5 from 1 to 100, but half of those are multiples of 2. The multiples of 5 which are not multiples of 2 are
  $$\{5, 15, 25, 35, 45, 55, 65, 75, 85, 95\}.$$
  We still need to remove numbers that are multiples of 3. After doing so we are left with
  $$\{5, 25, 35, 55, 65, 85, 95\}.$$
  So there are 7 numbers from 1 to 100 that are multiples of 5 but not 2, 3 or 4.

Therefore, once he has finished, E Z Dealer is left with $100 - (7 + 2 + 4 + 14 + 14 + 7) = 100 - 48 = 52$ cards with the red side facing up.

Extension: Suppose E Z Dealer continues flipping cards in this manner. So, after he has flipped all cards whose number is a multiple of 5, he then flips all cards whose card number is a multiple of 6, then 7, then 8, and so on until he flips all cards whose number is a multiple of 100. Once E Z has finished, how many cards will have the red side facing up?