Problem of the Week
## Problem E and Solution
The King's Coach

Problem

A king loves to travel from his castle to his summer house by coach. He orders his coachman to never go straight when he arrives at an intersection and to never travel along the same section of road twice during a trip. The coachman must turn either right or left when he comes to any intersection. The map shows all of the roads leading from the palace to the summer house. The thick black lines indicate roads. All roads connecting two adjacent intersections are 1 km long. The coachman wants to take the shortest route. Determine the length of the shortest route and justify why no shorter route is possible.

Solution

To get from the castle (bottom left corner) to the summer house (top right corner), the coach must pass through at least one of the four numbered roads on the adjacent diagram.

In the solution, we will examine four cases: routes which take us through each of the four numbered roads. A diagram will be presented for each possibility.
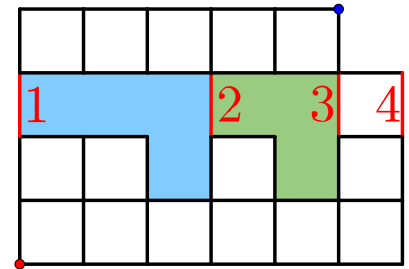
On each diagram, the start point will be labeled C, the endpoint will be labeled H, the southend of the road used in a particular case will be labeled S, and the north end will be labeled N.

Some details, for the sake of brevity, will be omitted from the solution and left for the solver to consider further. In the solution, turns will be described in terms of north, south, east and west.

1. **What is the length of the shortest path if the route passes through road 1?**

   If we travel from S to N, travel on the north-south roads immediately above and below would violate the king's order. On the diagram there is an X on each of these roads. However, to travel along S to N there are roads which must be traveled on. These are marked on the diagram with a ✓. The route shown on the diagram is the shortest route. This route is 15 km long.
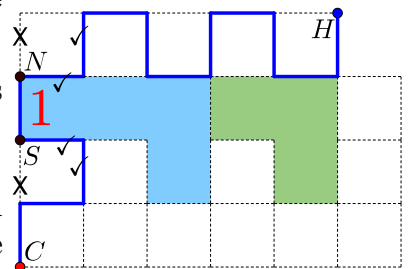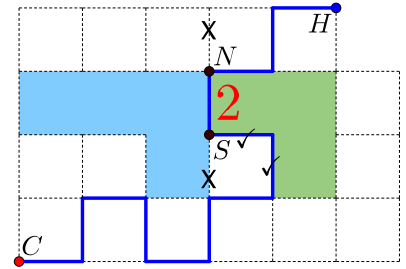
In the part of the route from C to S, the shortest route is 4 km and goes north - east - north - west. It would also be possible to extend this route by going north - east - south - east - north - west - north - west, but this route is clearly longer. There is no route starting from C that goes east first which is able to legally pass through road 1. In traveling from N to H, the shortest route is shown and is 10 km long. At some points along the route from N to H, alternate choices can be made but these choices lead to invalid situations or lengthen the route. The solver may wish to confirm this.

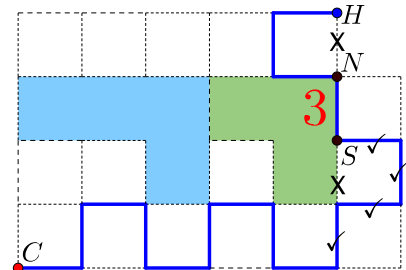2. **What is the length of the shortest path if the route passes through road 2?**

   If we travel from S to N, travel on the north-south roads immediately above and below would violate the king's order. On the diagram there is an X on each of these roads. However, to travel along S to N there are roads which must be traveled on. These are marked on the diagram with a ✓. The route shown on the diagram is the shortest route. This route is 13 km long.

In the part of the route from C to S, the shortest route is 9 km. In traveling from N to H, the shortest route is 3 km long. At some points along the route from C to S or N to H, alternate choices can be made but these choices either lead to invalid situations or lengthen the route. The solver may wish to confirm this.

3. **What is the length of the shortest path if the route passes through road 3?**

   If we travel from S to N, travel on the north-south roads immediately above and below would violate the king's order. On the diagram there is an X on each of these roads. However, to travel along S to N there are roads which must be traveled on. These are marked on the diagram with a ✓. The route shown on the diagram is the shortest route. This route is 17 km long.

In the part of the route from C to S, the shortest route is 13 km. In traveling from N to H, the shortest route is 3 km long. At some points along the route from C to S, alternate choices can be made but these choices either lead to invalid situations or lengthen the route. At two intersections along the route from N to H, alternate choices can be made but these choices would both lead to invalid situations. The solver may wish to confirm this.

4. **What is the length of the shortest path if the route passes through road 4?**

   If we travel from S to N, travel on the north-south roads immediately above and below would violate the king's order. On the diagram there is an X on each of these roads. However, to travel along S to N there are roads which must be traveled on. These are marked on the diagram with a ✓. The route shown on the diagram is the shortest route. This route is 15 km long.
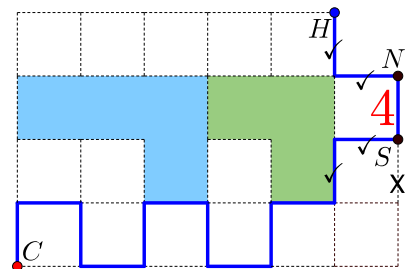
In the part of the route from C to S, the shortest route is 12 km. In traveling from N to H, the shortest route is 2 km long. At some points along the route from C to S, alternate choices can be made but these choices either lead to invalid situations, lengthen the route or have the same length. The solver may wish to confirm this. The route from N to H is clearly the shortest one possible.

After examining the four possible cases, the shortest route is 13 km and passes through road 2.