



Problem of the Week

Problem E and Solution

The Chances of This?

Problem

As part of their annual training, Santa’s eight reindeer, Dasher, Dancer, Prancer, Vixen, Comet, Cupid, Donner, Blitzen, participate in reindeer games. Rudolph was not allowed to play in any reindeer games. In one of the games, the reindeer must arrange themselves in a line. There are a few rules that the reindeer must follow. First, reindeer with the letter *r* in their name cannot stand together in adjacent positions in the line. Second, since Donner and Blitzen look so much alike, they can never stand in adjacent positions in the line. The reindeer randomly organize themselves in the line. What is the probability that they arrange themselves in a correct order?


Solution

Diagrams will be presented before an explanation of each case. Let Donner’s position in the line be marked with a *D*. Let the the other reindeer with an *r* in their name have their positions marked with an *R*. If a position has a number in it, that will represent the number of ways that position can be filled (by reindeer without *r* in their name and not Blitzen).

Four of the eight reindeer have an *r* in their name. The problem is complicated by the fact that Donner and Blitzen cannot stand together. We will break the problem into two cases.

1. Donner is in position 1 or position 8.

	D	3	R	_	R	_	R	_
	D	3	R	_	R	_	_	R
	D	3	R	_	_	R	_	R
	D	3	_	R	_	R	_	R
	_	R	_	R	_	R	3	D
	R	_	_	R	_	R	3	D
	R	_	R	_	_	R	3	D
	R	_	R	_	R	_	3	D



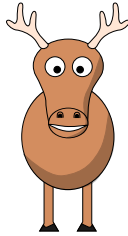
There are 8 configurations (shown above) with Donner in the first or last spot. For each of these, there are 3 possible ways for the reindeer to fill in the empty spot immediately adjacent to Donner (these reindeer do not have an *r* in their name, nor are they Blitzen). For each of these, there are $(3 \times 2 \times 1)$ ways to place the other reindeer with *r* in their name and $(3 \times 2 \times 1)$ ways to place the remaining reindeer, including Blitzen.

There are $8 \times 3 \times (3 \times 2 \times 1) \times (3 \times 2 \times 1) = 864$ ways to place the reindeer properly with Donner in the first or last spot.





2. Donner is in positions 2 through 7.



3	<i>D</i>	2	<i>R</i>	_	<i>R</i>	_	<i>R</i>
<i>R</i>	3	<i>D</i>	2	_	<i>R</i>	_	<i>R</i>
<i>R</i>	3	<i>D</i>	2	<i>R</i>	_	<i>R</i>	_
<i>R</i>	3	<i>D</i>	2	<i>R</i>	_	_	<i>R</i>
_	<i>R</i>	3	<i>D</i>	2	<i>R</i>	_	<i>R</i>
<i>R</i>	_	3	<i>D</i>	2	<i>R</i>	_	<i>R</i>
<i>R</i>	_	<i>R</i>	3	<i>D</i>	2	<i>R</i>	_
<i>R</i>	_	<i>R</i>	3	<i>D</i>	2	_	<i>R</i>
<i>R</i>	_	<i>R</i>	_	3	<i>D</i>	2	<i>R</i>
_	<i>R</i>	_	<i>R</i>	3	<i>D</i>	2	<i>R</i>
<i>R</i>	_	_	<i>R</i>	3	<i>D</i>	2	<i>R</i>
<i>R</i>	_	<i>R</i>	_	<i>R</i>	2	<i>D</i>	3



There are 12 configurations (shown above) with Donner in one of the spots from position 2 to position 7. For each of these, there are 3 possible ways for the reindeer to fill in the first empty spot immediately adjacent to Donner and two ways to fill the other empty spot immediately adjacent to Donner (these reindeer do not have an *r* in their name, nor are they Blitzen). For each of these, there are $(3 \times 2 \times 1)$ ways to place the reindeer with *r* in their name and (2×1) ways to place the remaining reindeer, including Blitzen.

There are $12 \times (3 \times 2) \times (3 \times 2 \times 1) \times (2 \times 1) = 864$ ways to place the reindeer properly with Donner in one of the spots 2 through 7.

The cases have no overlapping possibilities and we have considered all of the possible placements of Donner. Therefore, there are $864 + 864 = 1728$ ways for the reindeer to line up correctly.

If the reindeer could stand in any position in the line, the number of possible ways to line up is

$$8 \times 7 \times 6 \times \cdots \times 3 \times 2 \times 1 = 40\,320.$$

The probability of the reindeer randomly lining up in a correct formation is

$$\frac{1\,728}{40\,320} = \frac{3}{70} \doteq 0.043. \text{ There is a 4.3\% chance of the reindeer lining up correctly.}$$

The chances are extremely slight that the reindeer will line up randomly and be ordered correctly.

