Problem of the Week

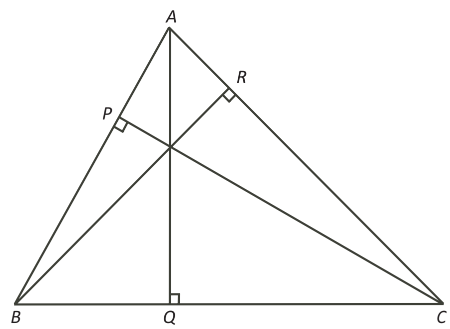
Problem E and Solution

From Altitudes to Angles to Sides

Problem

In $\triangle ABC$, $\angle BAC$ is the largest angle and $\angle ACB$ is the smallest angle. AQ, BR, and CP are altitudes with lengths 21 cm, 24 cm, and 56 cm, respectively.

Determine the size of $\angle ABC$ and the lengths of the sides of $\triangle ABC$.



Solution

Let $BC = a$, $AC = b$ and $AB = c$.

We can find the area of the triangle by multiplying the length of the altitude (the height) by the corresponding base and dividing by 2. Therefore,

$$\frac{AQ \times BC}{2} = \frac{BR \times AC}{2} = \frac{CP \times AB}{2}$$

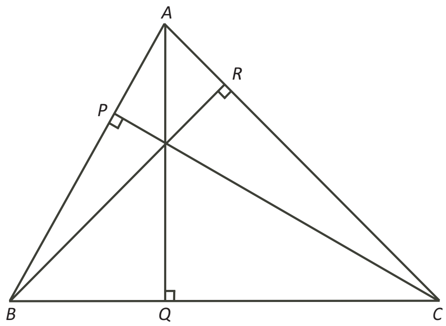
But $AQ = 21$, $BC = a$, $BR = 24$, $AC = b$, $CP = 56$, and $AB = c$. Multiplying through by 2 and substituting we obtain

$$21a = 24b = 56c.$$

From $21a = 24b$ we obtain $b = \frac{21}{24}a = \frac{7}{8}a$ and from $21a = 56c$ we obtain $c = \frac{21}{56}a = \frac{3}{8}a$. The ratio of the sides in $\triangle ABC$ is $a : b : c = a : \frac{7}{8}a : \frac{3}{8}a = 8 : 7 : 3$. Let $BC = 8x$, $AC = 7x$, and $AB = 3x$, $x > 0$.

Using the cosine law,

$$\begin{aligned}
AC^2 &= AB^2 + CB^2 - 2(AB)(CB)\cos(\angle ABC) \\
(7x)^2 &= (3x)^2 + (8x)^2 - 2(3x)(8x)\cos(\angle ABC) \\
49x^2 &= 9x^2 + 64x^2 - 48x^2\cos(\angle ABC) \\
\text{Dividing by } x^2 \text{ since } x > 0, \quad 49 &= 73 - 48\cos(\angle ABC) \\
\text{Rearranging,} \quad 48\cos(\angle ABC) &= 24 \\
\cos(\angle ABC) &= \frac{1}{2} \\
\therefore \angle ABC &= 60^\circ
\end{aligned}$$



In right $\triangle BPC$,

$$\begin{aligned}
\frac{PC}{BC} &= \sin 60^\circ \\
BC &= \frac{PC}{\sin 60^\circ} \\
BC &= \frac{56}{\frac{\sqrt{3}}{2}} \\
BC &= \frac{112}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} \\
BC &= \frac{112\sqrt{3}}{3} \\
\text{But } BC &= 8x \\
\therefore 8x &= \frac{112\sqrt{3}}{3} \\
x &= \frac{14\sqrt{3}}{3} \\
3x &= 14\sqrt{3} \\
7x &= \frac{98\sqrt{3}}{3}
\end{aligned}$$

The side lengths of $\triangle ABC$ are $AB = 3x = 14\sqrt{3}$, $AC = 7x = \frac{98\sqrt{3}}{3}$ and $BC = \frac{112\sqrt{3}}{3}$.