Problem of the Week  
Problem E and Solution  
From Altitudes to Angles to Sides

Problem
In $\triangle ABC$, $\angle BAC$ is the largest angle and $\angle ACB$ is the smallest angle. $AQ$, $BR$, and $CP$ are altitudes with lengths 21 cm, 24 cm, and 56 cm, respectively.

Determine the size of $\angle ABC$ and the lengths of the sides of $\triangle ABC$.

Solution
Let $BC = a$, $AC = b$ and $AB = c$.

We can find the area of the triangle by multiplying the length of the altitude (the height) by the corresponding base and dividing by 2. Therefore,

$$\frac{AQ \times BC}{2} = \frac{BR \times AC}{2} = \frac{CP \times AB}{2}$$

But $AQ = 21$, $BC = a$, $BR = 24$, $AC = b$, $CP = 56$, and $AB = c$. Multiplying through by 2 and substituting we obtain $21a = 24b = 56c$.

From $21a = 24b$ we obtain $b = \frac{21}{24}a = \frac{7}{8}a$ and from $21a = 56c$ we obtain $c = \frac{21}{56}a = \frac{3}{8}a$. The ratio of the sides in $\triangle ABC$ is $a : b : c = a : \frac{7}{8}a : \frac{3}{8}a = 8 : 7 : 3$. Let $BC = 8x$, $AC = 7x$, and $AB = 3x$, $x > 0$.

Using the cosine law,

$$AC^2 = AB^2 + CB^2 - 2(AB)(CB) \cos(\angle ABC)$$

$$(7x)^2 = (3x)^2 + (8x)^2 - 2(3x)(8x) \cos(\angle ABC)$$

$$49x^2 = 9x^2 + 64x^2 - 48x^2 \cos(\angle ABC)$$

Dividing by $x^2$ since $x > 0$,

$$49 = 73 - 48 \cos(\angle ABC)$$

Rearranging,

$$48 \cos(\angle ABC) = 24$$

$$\cos(\angle ABC) = \frac{1}{2}$$

$\therefore \angle ABC = 60^\circ$
In right $\triangle BPC$,

$$
\frac{PC}{BC} = \sin 60^\circ
$$

$$
BC = \frac{PC}{\sin 60^\circ}
$$

$$
BC = \frac{56}{\sqrt{3}}
$$

$$
BC = \frac{112\sqrt{3}}{3}
$$

But $BC = 8x$

$$
\therefore 8x = \frac{112\sqrt{3}}{3}
$$

$$
x = \frac{14\sqrt{3}}{3}
$$

$$
3x = 14\sqrt{3}
$$

$$
7x = \frac{98\sqrt{3}}{3}
$$

The side lengths of $\triangle ABC$ are $AB = 3x = 14\sqrt{3}$, $AC = 7x = \frac{98\sqrt{3}}{3}$ and $BC = \frac{112\sqrt{3}}{3}$. 