



Problem of the Week

Problem E and Solution

Putting the Parts Together

Problem

Determine the number of solutions to

$$\frac{P}{Q} - \frac{Q}{P} = \frac{P+Q}{PQ}$$

where P and Q are both integers, $-9 \leq P \leq 9$ and $-9 \leq Q \leq 9$.

Solution

$$\begin{aligned} \frac{P}{Q} - \frac{Q}{P} &= \frac{P+Q}{PQ} \\ \frac{P^2}{PQ} - \frac{Q^2}{PQ} &= \frac{P+Q}{PQ} && \text{Common Denominator} \\ \frac{P^2 - Q^2}{PQ} &= \frac{P+Q}{PQ} && \text{Simplify} \\ \frac{(P-Q)(P+Q)}{PQ} &= \frac{(1)(P+Q)}{PQ} && \text{Factor Left Side Numerator} \end{aligned}$$

Since the two sides are equal, $P - Q = 1$ or $P + Q = 0$. Also, P and Q cannot equal zero. Otherwise at least two of the denominators, P , Q , and PQ , would equal zero and division by zero is undefined. We will look at each possibility separately.

1. $P - Q = 1$, $P \neq 0$ and $Q \neq 0$.

In this case, we see that P and Q differ by 1 and $P > Q$. The largest value of P is 9. When $P = 9$, $Q = 8$. The smallest value of Q is -9 . When $Q = -9$, $P = -8$, a value which is 1 more than the value of Q . So P can take on all of the integer values from -8 to 9 except $P = 0$. But when $P = 1$, $Q = 0$. We would have to remove this value of P as well. There are 18 values for P from -8 to 9. After removing $P = 0$ and $P = 1$, there are 16 values for P and therefore 16 corresponding values for Q . The equation has 16 solutions such that $P - Q = 1$, P and Q are integers, $-9 \leq P \leq 9$ and $-9 \leq Q \leq 9$.

2. $P + Q = 0$, $P \neq 0$ and $Q \neq 0$.

In this case, $P + Q = 0$ or $P = -Q$. When $P = 9$, the largest integer value it can take on, $Q = -9$. Similarly, when $P = -9$, the smallest integer value it can take on, $Q = 9$. There are 19 integer values that P can take on but this includes $P = 0$, $Q = 0$. So the equation has $19 - 1 = 18$ solutions such that $P + Q = 0$, P and Q are integers, $-9 \leq P \leq 9$ and $-9 \leq Q \leq 9$.

We have considered all possible cases. Therefore, there are $16 + 18 = 34$ solutions to the equation.

