Problem of the Week
Problem E and Solution
Not That Kind of Median

Problem
A median is a line segment drawn from the vertex of a triangle to the midpoint of the opposite side. In \( \triangle ABC \), \( \angle ABC = 90^\circ \). A median is drawn from \( A \) meeting \( BC \) at \( M \) such that \( AM = 5 \). A second median is drawn from \( C \) meeting \( AB \) at \( N \) such that \( CN = 2\sqrt{10} \).

Determine the length of the longest side of \( \triangle ABC \).

Solution
Since \( AM \) is a median, \( M \) is the midpoint of \( BC \). Then \( BM = MC = y \).

Since \( CN \) is a median, \( N \) is the midpoint of \( AB \). Then \( AN = NB = x \).

\( \triangle NBC \) is right angled since \( \angle B = 90^\circ \). Using the Pythagorean Theorem,

\[
NB^2 + BC^2 = CN^2
\]
\[
x^2 + (2y)^2 = (2\sqrt{10})^2
\]
\[
x^2 + 4y^2 = 40 \quad (1)
\]

\( \triangle ABM \) is right angled since \( \angle B = 90^\circ \). Using the Pythagorean Theorem,

\[
AB^2 + BM^2 = AM^2
\]
\[
(2x)^2 + y^2 = 5^2
\]
\[
4x^2 + y^2 = 25 \quad (2)
\]

Adding (1) and (2), \( 5x^2 + 5y^2 = 65 \)

Dividing by 5, \( x^2 + y^2 = 13 \quad (3) \)

The longest side of \( \triangle ABC \) is the hypotenuse \( AC \). Using the Pythagorean Theorem,

\[
AC^2 = AB^2 + BC^2
\]
\[
= (2x)^2 + (2y)^2
\]
\[
= 4x^2 + 4y^2
\]
\[
= 4(x^2 + y^2)
\]

Substituting from (3) above, \( AC^2 = 4(13) \)

Taking the square root, \( AC = 2\sqrt{13} \)

\( \therefore \) the length of the longest side is \( 2\sqrt{13} \) units.

Note: The solver could actually solve a system of equations to find \( x = 2 \) and \( y = 3 \) and then proceed to solve for the longest side. The above approach was provided to expose the solver to alternate thinking about the solution of this problem.