Problem of the Week
Problem E and Solution
Balloons Away!

Problem
John has just built a balloon launcher. He is using coordinates and coordinate geometry to describe the location of the balloon.

John launches a balloon from point D on the ground. The balloon follows a perfect parabolic trajectory so that it passes through the point A(9, 8), then reaches its peak at E(7, 9), and finally passes through a hoop located at B(b, 5) before returning to the ground at point C and bursting. The ground between C and D is flat. Determine the area of quadrilateral ABCD.

Solution
We need to find the equation of the parabola. Then, in order to answer the question, we are required to find the x-intercepts of the parabola and the x-coordinate of point B on the parabola.

We are given the peak E(7, 9) so we have the vertex of the parabola. Using the vertex form of the equation of a parabola, \( y = a(x - h)^2 + k \), with vertex \((h, k) = (7, 9)\), the equation of the parabola looks like \( y = a(x - 7)^2 + 9 \).

The point A(9, 8) is on the parabola so we can substitute \((x, y) = (9, 8)\) into the equation \( y = a(x - 7)^2 + 9 \) to find \( a \).

\[
\begin{align*}
8 &= a(9 - 7)^2 + 9 \\
8 &= a(4) + 9 \\
-1 &= 4a \\
-\frac{1}{4} &= a
\end{align*}
\]

The equation of the parabola is \( y = -\frac{1}{4}(x - 7)^2 + 9 \).

To find the x-coordinate of B(b, 5), substitute \( y = 5 \) into the equation of the parabola.

\[
\begin{align*}
5 &= -\frac{1}{4}(b - 7)^2 + 9 \\
-4 &= -\frac{1}{4}(b - 7)^2 \\
16 &= (b - 7)^2 \\
\pm 4 &= b - 7
\end{align*}
\]

It follows that \( b - 7 = -4 \) or \( b - 7 = 4 \). Then \( b = 3 \) or \( b = 11 \). The point B is to the left of the vertex so \( b < 7 \). The coordinates of B are (3, 5).
To find the $x$-intercepts of the parabola, substitute $y = 0$ into the equation of the parabola.

\[
0 = -\frac{1}{4}(x - 7)^2 + 9 \\
-9 = -\frac{1}{4}(x - 7)^2 \\
36 = (x - 7)^2 \\
\pm 6 = x - 7
\]

It follows that $x - 7 = -6$ or $x - 7 = 6$. Then the $x$-intercepts of the parabola are 1 and 13. The point $C$ is to the left of the vertex and the point $D$ is to the right of the vertex. The coordinates of $C$ are $(1, 0)$ and $D$ are $(13, 0)$.

This information has been added to the graph. There are many ways to determine the area of $ABCD$.

From $B(3, 5)$ and $A(9, 8)$ drop perpendiculars to the $x$-axis, intersecting at $F(3, 0)$ and $G(9, 0)$, respectively. From $B(3, 5)$ draw a perpendicular to $AG$, intersecting at $H(9, 5)$. Draw line segment $BG$.

We will use the diagram to find the lengths of several horizontal and vertical line segments that will be required for the area calculation.

$BH = 9 - 3 = 6$, $CG = 9 - 1 = 8$, $GD = 13 - 9 = 4$, $BF = 5 - 0 = 5$, and $AG = 8 - 0 = 8$.

To determine the area, we will find the sum of the areas of $\triangle CGB$, $\triangle AGD$ and $\triangle AGB$.

\[
\text{Area } ABCD = \text{Area } \triangle CGB + \text{Area } \triangle AGD + \text{Area } \triangle AGB \\
= \frac{CG \times BF}{2} + \frac{AG \times GD}{2} + \frac{AG \times BH}{2} \\
= \frac{8 \times 5}{2} + \frac{8 \times 4}{2} + \frac{8 \times 6}{2} \\
= 20 + 16 + 24 \\
= 60 \text{ units}^2
\]

The area of $ABCD$ is 60 units$^2$. 