Problem of the Week

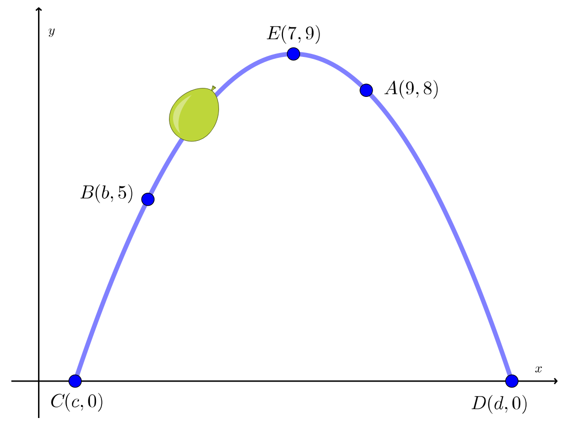
Problem E and Solution

Balloons Away!

Problem

John has just built a balloon launcher. He is using coordinates and coordinate geometry to describe the location of the balloon.

John launches a balloon from point D on the ground. The balloon follows a perfect parabolic trajectory so that it passes through the point $A(9, 8)$, then reaches its peak at $E(7, 9)$, and finally passes through a hoop located at $B(b, 5)$ before returning to the ground at point C and bursting. The ground between C and D is flat. Determine the area of quadrilateral $ABCD$.



Solution

We need to find the equation of the parabola. Then, in order to answer the question, we are required to find the x-intercepts of the parabola and the x-coordinate of point B on the parabola.

We are given the peak $E(7, 9)$ so we have the vertex of the parabola. Using the vertex form of the equation of a parabola, $y = a(x - h)^2 + k$, with vertex $(h, k) = (7, 9)$, the equation of the parabola looks like $y = a(x - 7)^2 + 9$.

The point $A(9, 8)$ is on the parabola so we can substitute $(x, y) = (9, 8)$ into the equation $y = a(x - 7)^2 + 9$ to find a.

$$\begin{aligned} 8 &= a(9-7)^2 + 9 \\ 8 &= a(4) + 9 \\ -1 &= 4a \\ -\frac{1}{4} &= a \end{aligned}$$

The equation of the parabola is $y = -\frac{1}{4}(x - 7)^2 + 9$.

To find the x-coordinate of $B(b, 5)$, substitute $y = 5$ into the equation of the parabola.

$$\begin{aligned} 5 &= -\frac{1}{4}(b-7)^2 + 9 \\ -4 &= -\frac{1}{4}(b-7)^2 \\ 16 &= (b-7)^2 \\ \pm 4 &= b - 7 \end{aligned}$$

It follows that $b - 7 = -4$ or $b - 7 = 4$. Then $b = 3$ or $b = 11$. The point B is to the left of the vertex so $b < 7$. The coordinates of B are $(3, 5)$.

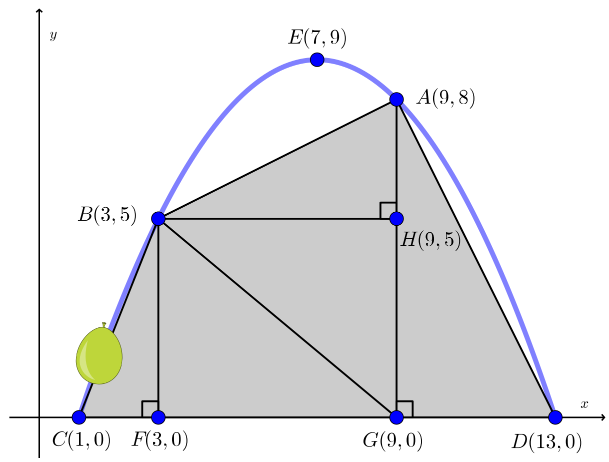
To find the x-intercepts of the parabola, substitute $y = 0$ into the equation of the parabola.

$$\begin{aligned} 0 &= -\frac{1}{4}(x-7)^2 + 9 \\ -9 &= -\frac{1}{4}(x-7)^2 \\ 36 &= (x-7)^2 \\ \pm 6 &= x - 7 \end{aligned}$$

It follows that $x - 7 = -6$ or $x - 7 = 6$. Then the x-intercepts of the parabola are 1 and 13. The point C is to the left of the vertex and the point D is to the right of the vertex. The coordinates of C are $(1, 0)$ and D are $(13, 0)$.

This information has been added to the graph. There are many ways to determine the area of $ABCD$.

From $B(3, 5)$ and $A(9, 8)$ drop perpendiculars to the x-axis, intersecting at $F(3, 0)$ and $G(9, 0)$, respectively. From $B(3, 5)$ draw a perpendicular to AG, intersecting at $H(9, 5)$. Draw line segment BG.



We will use the diagram to find the lengths of several horizontal and vertical line segments that will be required for the area calculation.

$BH = 9 - 3 = 6,\ CG = 9 - 1 = 8,\ GD = 13 - 9 = 4,\ BF = 5 - 0 = 5,$ and $AG = 8 - 0 = 8.$

To determine the area, we will find the sum of the areas of $\triangle CGB$, $\triangle AGD$ and $\triangle AGB$.

$$\begin{aligned} \text{Area } ABCD &= \text{Area } \triangle CGB + \text{Area } \triangle AGD + \text{Area } \triangle AGB \\ &= \frac{CG \times BF}{2} + \frac{AG \times GD}{2} + \frac{AG \times BH}{2} \\ &= \frac{8 \times 5}{2} + \frac{8 \times 4}{2} + \frac{8 \times 6}{2} \\ &= 20 + 16 + 24 \\ &= 60 \text{ units}^2 \end{aligned}$$

The area of $ABCD$ is 60 units ^{2}.