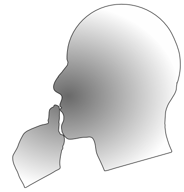
Problem of the Week
Problem E and Solution
This is Some Function



Problem

For some function $f(x) = ax^3 + bx^2 + cx + d$ where $a,\ b,\ c$ and d are integers, we know the following information: the y-intercept is 5, $f(2) = -3$, $40 < f(4) < 50$, and $240 < f(6) < 250$. Determine the value of $f(3)$.

Solution

We will process the information in the order that it was provided.

- The y-intercept is 5 so we know that $f(0) = 5$.

$$\begin{aligned} f(0) &= 5 \\ a(0)^3 + b(0)^2 + c(0) + d &= 5 \\ d &= 5 \end{aligned}$$

  The function is now $f(x) = ax^3 + bx^2 + cx + 5$.

- We are then given that $f(2) = -3$.

$$\begin{aligned} f(2) &= -3 \\ a(2)^3 + b(2)^2 + c(2) + 5 &= -3 \\ 8a + 4b + 2c + 5 &= -3 \\ 8a + 4b + 2c &= -8 \\ 4a + 2b + c &= -4 \\ c &= -4a - 2b - 4 \qquad (1) \end{aligned}$$

- Next, we know that $40 < f(4) < 50$. We will use the notes provided concerning the solving of inequalities.

$$\begin{array}{lrcccl} & 40 & < & f(4) & < & 50 \\ & 40 & < & a(4)^3 + b(4)^2 + c(4) + 5 & < & 50 \\ & 40 & < & 64a + 16b + 4c + 5 & < & 50 \\ \text{Substitute for } c \text{ from (1)} & 40 & < & 64a + 16b + 4(-4a - 2b - 4) + 5 & < & 50 \\ & 40 & < & 64a + 16b - 16a - 8b - 16 + 5 & < & 50 \\ & 40 & < & 48a + 8b - 11 & < & 50 \\ \text{Adding 11 to each part} & 51 & < & 48a + 8b & < & 61 \\ \text{Dividing each part by 8} & 6.375 & < & 6a + b & < & 7.625 \end{array}$$

  Both a and b are integers so $6a + b$ will also be an integer. The only integer greater than 6.375 and less than 7.625 is 7. Therefore, it follows that $6a + b = 7$. (2)

- The last piece of given information is $240 < f(6) < 250$. We will use the notes provided concerning the solving of inequalities.

$$\begin{array}{lrcccl}
 & 240 & < & f(6) & < & 250 \\
 & 240 & < & a(6)^3 + b(6)^2 + c(6) + 5 & < & 250 \\
 & 240 & < & 216a + 36b + 6c + 5 & < & 250 \\
\text{Substitute for } c \text{ from (1)} & 240 & < & 216a + 36b + 6(-4a - 2b - 4) + 5 & < & 250 \\
 & 240 & < & 216a + 36b - 24a - 12b - 24 + 5 & < & 250 \\
 & 240 & < & 192a + 24b - 19 & < & 250 \\
\text{Adding 19 to each part} & 259 & < & 192a + 24b & < & 269 \\
\text{Dividing each part by 24} & 10\frac{19}{24} & < & 8a + b & < & 11\frac{5}{24}
\end{array}$$

Both a and b are integers so $8a + b$ will also be an integer. The only integer greater than $10\frac{19}{24}$ and less than $11\frac{5}{24}$ is 11. Therefore, it follows that $8a + b = 11$. (3)

Now we have a system of equations:

$$\begin{array}{rcll}
6a + b & = & 7 & (2) \\
8a + b & = & 11 & (3)
\end{array}$$

By subtracting (2) from (3), we eliminate b obtaining $2a = 4$ and $a = 2$ follows. Substituting $a = 2$ in (2), we obtain $12 + b = 7$ and $b = -5$ follows.

Substituting $a = 2$ and $b = -5$ in (1)

$$\begin{array}{rcl}
c & = & -4a - 2b - 4 \\
 & = & -4(2) - 2(-5) - 4 \\
 & = & -8 + 10 - 4 \\
 & = & -2
\end{array}$$

Since $f(x) = ax^3 + bx^2 + cx + d$ with $a = 2$, $b = -5$, $c = -2$ and $d = 5$, then the function becomes $f(x) = 2x^3 - 5x^2 - 2x + 5$.

To find the value of $f(3)$, we substitute $x = 3$ into the function.

$$\begin{array}{rcl}
f(x) & = & 2x^3 - 5x^2 - 2x + 5 \\
f(3) & = & 2(3)^3 - 5(3)^2 - 2(3) + 5 \\
 & = & 2(27) - 5(9) - 6 + 5 \\
 & = & 54 - 45 - 6 + 5 \\
 & = & 8
\end{array}$$

Therefore, $f(3) = 8$.