Problem of the Week
Problem E and Solution
How Low Will It Go?

Problem
Suppose $y = 5x^2 + ax + b$, $a \neq b$, is a parabola that passes through the points $A(a, b)$ and $B(b, a)$. Determine the minimum value of the parabola.

Solution
Since $A(a, b)$ is on the parabola, it satisfies the equation of the parabola. We can substitute $x = a$ and $y = b$ into the equation $y = 5x^2 + ax + b$.


give us \[
b = 5a^2 + a^2 + b
\]
\[
b = 6a^2 + b
\]
\[
0 = 6a^2
\]
\[
0 = a^2
\]
\[
0 = a
\]

The equation becomes $y = 5x^2 + 0x + b$ or simply $y = 5x^2 + b$.

Since $B(b, a)$ is on the parabola, it satisfies the equation of the parabola. We can substitute $x = b$ and $y = a = 0$ into the equation $y = 5x^2 + b$.


give us \[
0 = 5b^2 + b
\]
\[
0 = b(5b + 1)
\]
\[
b = 0 \quad \text{or} \quad 5b + 1 = 0
\]
\[
\begin{align*}
b &= -\frac{1}{5}
\end{align*}
\]

Since $a \neq b$ and $a = 0$, then $b = 0$ is inadmissible. Therefore, $b = -\frac{1}{5}$ and the equation becomes $y = 5x^2 - \frac{1}{5}$. The vertex of the parabola is $(0, -\frac{1}{5})$ and so the minimum value is $-\frac{1}{5}$. 