



Problem of the Week

Problem E and Solution

Keys to Success

Problem

The Fine Arts High School Piano Club has 30 student members, some from Grade 11 and the remainder from Grade 12. Each pair of students from the club must play a piano duet together over the course of the year. When two grade 11 students play together, they need 2 hours of practice time. When a grade 11 and a grade 12 student play together, they need 3 hours of practice time. When two grade 12 students play together, they need 4 hours of practice time. In total, the students need 1392 hours of practice time. How many of the 30 members of the club are Grade 11 students?

Solution

If 3 students $\{A, B, C\}$ are in the same grade, then there will be $3 \times 2 \div 2 = 3$ duet pairings, namely $\{AB\}$, $\{AC\}$, and $\{BC\}$. (If we look at this using a counting argument, there would be 3 choices for the first student and for each of these choices, there would be 2 choices for the second student, a total of $3 \times 2 = 6$ pairings, namely $\{AB\}$, $\{AC\}$, $\{BA\}$, $\{BC\}$, $\{CA\}$, and $\{CB\}$. Each pairing appears twice. Since order is not important we must divide by 2, getting us 3 possible pairings.)

If 3 students $\{A, B, C\}$ are in one grade and 2 students $\{D, E\}$ are in the other grade, then there will be $3 \times 2 = 6$ duet pairings, namely $\{AD\}$, $\{AE\}$, $\{BD\}$, $\{BE\}$, $\{CD\}$, and $\{CE\}$. (There are 3 choices for the grade 11 student in the pairing and for each of these choices, there are 2 possibilities for the choice of the grade 12 student. This gives a total of $3 \times 2 = 6$ pairings.)

This specific argument will now be applied to the general case.

Now, let a represent the number of Grade 11 students in the club and $(30 - a)$ represent the number of Grade 12 students in the club.





In general, since there are a students in Grade 11 and each must play a duet with every other student in Grade 11, there will be $a \times (a - 1) \div 2$ duets involving only Grade 11 students. Similarly, since there are $(30 - a)$ students in Grade 12 and each must play a duet with every other student in Grade 12, there will be $(30 - a) \times (30 - a - 1) \div 2 = (30 - a) \times (29 - a) \div 2$ duets involving only Grade 12 students. Since every Grade 11 student must play a duet with every Grade 12 student, there will be $a \times (30 - a)$ duets involving one student from each grade.

To determine the total amount of practice time required, take the number of students in each type of pairing and multiply by the number of hours of practice time required for each pairing type.

$$\begin{aligned} \text{Total Time} &= \text{Time for Grade 11 Pairs} + \text{Time for Grade 12 Pairs} \\ &\quad + \text{Time for Grade 11/12 Pairs} \end{aligned}$$

$$1392 = 2 \left[\frac{a \times (a - 1)}{2} \right] + 4 \left[\frac{(30 - a) \times (29 - a)}{2} \right] + 3[a \times (30 - a)]$$

$$1392 = a^2 - a + 2(870 - 59a + a^2) + 3(30a - a^2)$$

$$1392 = a^2 - a + 1740 - 118a + 2a^2 + 90a - 3a^2$$

$$1392 = -29a + 1740$$

$$29a = 348$$

$$a = 12$$

Therefore, 12 of the students in the club are in Grade 11.

