# Problem of the Week
## Problem D and Solution
## Grouping Digits

### Problem

Using only the digits 1, 2, 3, 4, and 5, a sequence is created as follows: one 1, two 2's, three 3's, four 4's, five 5's, six 1's, seven 2's, and so on. The sequence appears as

$$1,\ 2,\ 2,\ 3,\ 3,\ 3,\ 4,\ 4,\ 4,\ 4,\ 5,\ 5,\ 5,\ 5,\ 5,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ \cdots .$$

Determine the $2017^{\text{th}}$ digit in the sequence.

## Solution

The first group in the sequence contains one 1. The second group in the sequence contains two 2's. To the end of the second group of digits, there is a total of $1 + 2 = 3$ digits. The third group in the sequence contains three 3's. To the end of the third group of digits, there is a total of $1 + 2 + 3 = 6$ digits. The $n$th group in the sequence contains $n$ digits. To the end of the $n$th group of digits, there is a total of $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ digits.

So how many groups of digits are required for there to be at least 2017 digits in the sequence?

We need to find the value of $n$ so that $1 + 2 + 3 + \cdots + n \geq 2017$ and $1 + 2 + 3 + \cdots + (n - 1) < 2017$. At this point we will use trial and error. (On the next page, a more algebraic approach to finding the value of $n$ using the quadratic formula is presented.)

Suppose $n = 100$. Then $1 + 2 + 3 + \cdots + 100 = \frac{100(101)}{2} = 5050 > 2017$.

Suppose $n = 50$. Then $1 + 2 + 3 + \cdots + 50 = \frac{50(51)}{2} = 1275 < 2017$.

Suppose $n = 60$. Then $1 + 2 + 3 + \cdots + 60 = \frac{60(61)}{2} = 1830 < 2017$.

Suppose $n = 65$. Then $1 + 2 + 3 + \cdots + 65 = \frac{65(66)}{2} = 2145 > 2017$.

Suppose $n = 63$. Then $1 + 2 + 3 + \cdots + 63 = \frac{63(64)}{2} = 2016 < 2017$.

The $2017^{\text{th}}$ digit is the first number in the next group of digits. That is, the $2017^{\text{th}}$ digit is a digit in the $64^{\text{th}}$ group of digits.

So what is the digit in the $2017^{\text{th}}$ position? Since we cycle through the digits and there are only five digits used,

groups $6, 11, 16, \cdots, 56, 61, 66, \cdots$ will contain only 1's,
groups $7, 12, 17, \cdots, 57, 62, 67, \cdots$ will contain only 2's,
groups $8, 13, 18, \cdots, 58, 63, 68, \cdots$ will contain only 3's,
groups $9, 14, 19, \cdots, 59, 64, 69, \cdots$ will contain only 4's, and
groups $10, 15, 20, \cdots, 60, 65, 70, \cdots$ will contain only 5's.

Since the $2017^{\text{th}}$ digit is in the $64^{\text{th}}$ group of digits, it follows that the $2017^{\text{th}}$ digit is a 4.

Finding the value of $n$ algebraically

We will find the value of $n, n > 0$ so that

$$\begin{aligned} \frac{n(n+1)}{2} &= 2017 \\ n(n+1) &= 4034 \\ n^2 + n - 4034 &= 0 \end{aligned}$$

The quadratic formula can be used to solve for $n$.

$$\begin{aligned} n &= \frac{-1 \pm \sqrt{1^2 - 4(1)(-4034)}}{2} \\ &= \frac{-1 \pm \sqrt{16\,137}}{2} \end{aligned}$$

Since $n = \frac{-1-\sqrt{16\,137}}{2} < 0$, it is inadmissible.

Then $n = \frac{-1+\sqrt{16\,137}}{2} \approx 63.02$. But $n$ is an integer. So, interpreting our result, when $n = 63$, the sum $1 + 2 + 3 + \cdots + 63 < 2017$ and when $1 + 2 + 3 + \cdots + 64 > 2017$. Thus, the $2017^{\text{th}}$ digit is in the $64^{\text{th}}$ group of digits. It follows from the previous page that this digit a 4.