Problem of the Week
Problem D and Solution
It Can't be Done

Problem

Some two-digit positive integers cannot be expressed as the sum of two primes. It just can't be done. How many of these two-digit positive integers are there?

Solution

We will determine how many two-digit numbers can be written as the sum of two primes and then subtract that result from the total number of two-digit numbers.

There are 90 two-digit numbers, 45 of them are even and 45 of them are odd. From Goldbach's conjecture we already know that all 45 even two-digit numbers can be written as the sum of two prime numbers.

How many of the odd two-digit numbers can be written as the sum of two primes.

To start, it would be helpful to list all primes under 100:

2	3	5	7	11	13	17	19	23	29	31	37	41
43	47	53	59	61	67	71	73	79	83	89	97	

As you can see there are 25 prime numbers less than 100.

(For those of you wanting to find an interesting way to generate these primes, you might consider doing an online search of "The Sieve of Eratosthenes". Wikipedia has a demonstration of how this process works.)

In order to generate an odd number as the sum of two numbers, one of the numbers must be even and the other must be odd. Since 2 is the only even prime number it must be one of the two numbers used in the sum that generates the odd two-digit number. If q is the other prime number and q is odd, then $11 \leq 2 + q \leq 99$ which simplifies to $9 \leq q \leq 97$. From our list of prime numbers above, there are 21 prime numbers that satisfy this condition. Therefore, 21 odd two-digit numbers and 45 even two-digit numbers can be written as the sum of two primes. That is, $21 + 45 = 66$ two-digit numbers can be written as the sum of two prime numbers.

Therefore, $90 - 66 = 24$ two-digit numbers cannot be written as the sum of two prime numbers.

On the following page, the solution is illustrated with the use of charts.

To visualize the solution, recall that every even number greater than 2 can be expressed as the sum of 2 primes. There are 45 even two-digit numbers. We eliminate them from the table leaving the 45 odd two-digit numbers shown.

11	13	15	17	19
21	23	25	27	29
31	33	35	37	39
41	43	45	47	49
51	53	55	57	59
61	63	65	67	69
71	73	75	77	79
81	83	85	87	89
91	93	95	97	99

The only way to create an odd two-digit number using two primes is for one of the primes to be even and the other prime to be odd. Therefore, 2 must be one of the primes. From the solution on the previous page we determined that the odd primes must come from the primes greater than or equal to 9 but less than or equal to 97. There are 21 primes in this interval. When 2 is added to each of these primes, we find 21 odd two-digit numbers that can be expressed as the sum of two distinct primes. These numbers are shown in red in the following table.

11	13	15	17	19
21	23	25	27	29
31	33	35	37	39
41	43	45	47	49
51	53	55	57	59
61	63	65	67	69
71	73	75	77	79
81	83	85	87	89
91	93	95	97	99

When the 21 red numbers are removed from the table, the remaining 24 numbers are the two-digit numbers that cannot be expressed as the sum of two primes. They are shown in the final table below.

11			17	
	23		27	29
		35	37	
41			47	
51	53		57	59
		65	67	
71			77	79
	83		87	89
	93	95	97	