Problem of the Week

Problem D and Solution

More Coal Please

Problem

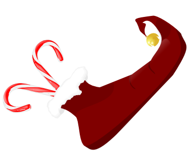
Santa fills the elf-shoes of the good elves with candy and other good things. However, the elf-shoes of the naughty elves generally receive some coal as well. Three of Santa's elves, the triplets, Zeta, Eta and Theta, have been very naughty this year. Santa is giving them a total of ten lumps of coal between them in their elf-shoes. Each of them must get at least one lump of coal. In how many different ways can Santa distribute the coal to Zeta, Eta and Theta?

Solution

We could completely list all of the possibilities but that would not be practical if there were more lumps of coal. We know that there are 10 lumps of coal and that each elf must receive at least one. We will consider a few cases to see if there is a pattern.

1. Zeta receives one lump of coal. Then Eta and Theta receive a total of $10 - 1 = 9$ lumps between them. This can be done in 8 possible ways:
$$\{(1,8),(2,7),(3,6),(4,5),(5,4),(6,3),(7,2),(8,1)\}.$$
2. Zeta receives two lumps of coal. Then Eta and Theta receive a total of $10 - 2 = 8$ lumps between them. This can be done in 7 possible ways:
$$\{(1,7),(2,6),(3,5),(4,4),(5,3),(6,2),(7,1)\}.$$
3. Zeta receives three lumps of coal. Then Eta and Theta receive a total of $10 - 3 = 7$ lumps between them. This can be done in 6 possible ways:
$$\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}.$$

We could continue but the pattern is established. We will summarize in the following table.



# of lumps for Zeta	# of lumps remaining	# of ways to distribute the remaining lumps between Eta and Theta
1	9	8
2	8	7
3	7	6
4	6	5
5	5	4
6	4	3
7	3	2
8	2	1

The total number of ways Santa can distribute 10 lumps of coal between the three naughty elves so that each receives at least one lump is $8+7+6+5+4+3+2+1 = 36$ ways. This sum can be computed by adding the positive integers from 1 to 8. However, it is also known that the sum of the first n positive integers can be calculated using the formula $\frac{n(n+1)}{2}$. In this case $n = 8$ so the sum is $\frac{8(9)}{2} = 36$.