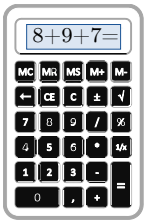
Problem of the Week
Problem D and Solution
Some Digit Sums



Problem

The *digit sum* of a number is found by, first, summing its digits. If the sum is greater than 9, then the digits of the sum are added. This process is repeated until a single digit number is obtained. The digit sum of 602 is 8 since $6 + 0 + 2 = 8$, and 8 is a single digit number. The digit sum of 897 is 6. However, it takes two steps to reach this sum. First, $8 + 9 + 7 = 24$, which is not a single digit number. Second, $2 + 4 = 6$, which is a single digit number and the process stops after the two steps. How many three-digit numbers have a digit sum of 5 that is reached in three or fewer steps?

Solution

We will consider three cases for the three-digit numbers: a digit sum of 5 in one step, a digit sum of 5 in two steps and a digit sum of 5 in three steps.

1. A digit sum of 5 that is reached in one step.

   Since the digit sum is 5, then no digit in the three-digit number can be higher than 5. It is straight forward to generate a list of all of the possible numbers.

104	113	122	131	140
203	212	221	230	
302	311	320		
401	410			
500				

   There are $5 + 4 + 3 + 2 + 1 = 15$ three-digit numbers with a digit sum of 5 that can reached in exactly one step.

2. A digit sum of 5 that is reached in two steps.

   The maximum sum of the digits of a three-digit number is $9 + 9 + 9 = 27$. In order to reach a digit sum of 5 in two steps, the initial sum must be a two digit number less than 28 whose digits sum to 5. There are only 2 two-digit numbers that satisfy this condition, namely 14 and 23.

If the sum of the digits of the three-digit number is 14, we can systematically generate the possible numbers. For example, if the first digit is 1, then the other two digits add to 13. This can be done with the digits 49, 58, 67, 76, 85, and 94. The 5 three-digit numbers with first digit 1 are shown in the first row of the following table. The remaining rows are generated in a similar manner.

149	158	167	176	185	194				
239	248	257	266	275	284	293			
329	338	347	356	365	374	383	392		
419	428	437	446	455	464	473	482	491	
509	518	527	536	545	554	563	572	581	590
608	617	626	635	644	653	662	671	680	
707	716	725	734	743	752	761	770		
806	815	824	833	842	851	860			
905	914	923	932	941	950				

There are $6 + 7 + 8 + 9 + 10 + 9 + 8 + 7 + 6 = 70$ three-digit numbers with a digit sum of 5 that can reached in exactly two steps with the initial sum of 14.

If the sum of the digits of the three-digit number is 23, we can systematically generate the possible numbers. If the final two digits of the three-digit number are 9 and 9, the first digit must be a 5. Therefore, no three-digit number less than 599 has digits that sum to 23. (For example if one of the digits is 4, then the sum of the other two digits must be 19 and this is impossible using two single digits.) The table is generated in a similar way to the one shown above.

599				
689	698			
779	788	797		
869	878	887	896	
959	968	977	986	995

There are $1 + 2 + 3 + 4 + 5 = 15$ three-digit numbers with a digit sum of 5 that can reached in exactly two steps with the initial sum of 23.

In total, there are $70 + 15 = 85$ three-digit numbers with a digit sum of 5 that can be reached in exactly two steps.

3. A digit sum of 5 that is reached in three steps.

   The maximum sum of the digits of a three-digit number is 27. The only number from 10 to 27 whose digits add to a two-digit number is 19. Its digit sum would then be 1, not 5. No three-digit number exists that has a digit sum of 5 reached in exactly 3 steps.

The above cases represent the only possible ways for a three-digit number to have a digit sum of 5 in three or fewer steps. In total there are $15 + 85 + 0 = 100$ three-digit numbers with a digit sum of 5, that can be reached in three or fewer steps.