Problem of the Week
Problem D and Solution
Another way to Add 'em Up

Problem

The number 1000 can be written as the sum of 16 consecutive positive integers. That is,

$1000 = 55 + 56 + 57 + 58 + 59 + 60 + 61 + 62 + 63 + 64 + 65 + 66 + 67 + 68 + 69 + 70.$

It is also possible to write 1000 as a sum of 25 consecutive positive integers. This is the maximum number of consecutive positive integers that could be used to create the sum. Determine the smallest of the positive integers in this sum.

Solution

Solution 1

Let n, $n+1$, $n+2$, $\cdots$, $n+23$, $n+24$ represent the 25 consecutive positive integers. Then,

$$\begin{aligned} n+n+1+n+2+\ \cdots\ +n+23+n+24 &= 1000 \\ 25n+(1+2+3+\ \cdots\ +23+24) &= 1000 \\ 25n+300 &= 1000 \qquad \text{(See note below.)} \\ 25n &= 700 \\ n &= 28 \end{aligned}$$

$\therefore$ the smallest number in the sum of 25 consecutive positive integers is 28.

Note:

The sum $1+2+3+\ \cdots\ +23+24$ can be calculated in a variety of ways. It is an arithmetic series with first term $a=1$, common difference $d=1$, and the number of terms $n=24$.

Using $S_n = \dfrac{n}{2}\left[2a+(n-1)d\right]$, $S_{24} = \dfrac{24}{2}\left[2(1)+(23)(1)\right] = 12(25) = 300$.

It is also known that the sum of the first n natural numbers can be calculated using the formula $\dfrac{n(n+1)}{2}$. Using the formula with $n=24$, the sum is $\dfrac{24\times 25}{2} = 300$.

Solution 2

Let n represent the middle integer of the 25 consecutive positive integers. Then there are 12 integers below the middle integer with the smallest integer being $(n-12)$ and 12 integers above the middle integer with the largest integer being $(n+12)$.

$$\begin{aligned}(n-12)+(n-11)+\cdots+(n-2)+(n-1)+n&\\ +(n+1)+(n+2)+\cdots+(n+11)+(n+12) &= 1000\\ 25n &= 1000 \qquad \text{(See note below.)}\\ n &= 40\\ n-12 &= 28\end{aligned}$$

$\therefore$ the smallest integer in the sum of 25 consecutive positive integers is 28.

Note:

The sum simplifies to $25n = 1000$ because for each positive integer 1 to 12 in the sum, the corresponding integer of opposite sign -1 to -12 also appears.

Then $(1+2+\cdots+11+12)+(-1-2-\cdots-11-12)=0$.

Solution 3

In this problem, we want to express 1000 as the sum of 25 consecutive positive integers. The average, $1000 \div 25 = 40$, is an integer. There will be twelve consecutive integers above the average and twelve consecutive integers below the average. The smallest integer will therefore be $40-12=28$.

Solution 2 is actually a mathematical justification of this result.