Problem of the Week
Problem D and Solution
Three by Three

Problem
The first 9 positive odd integers are placed in the $3 \times 3$ grid shown to the right in such a way that the sum of each row, column and diagonal is the same. Four of the numbers are shown and the other five numbers are hidden behind the letters $A$, $B$, $C$, $D$, and $E$. Determine the value of $A + E$.

Solution

Solution 1
The numbers to be placed in the table are 1, 3, 5, 7, 9, 11, 13, 15, and 17, the first 9 positive odd integers. The sum of all the numbers in the table is $1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 = 81$. It follows that the sum of the sums of the three rows is 81. But each row has the same sum so the sum of each row is $81 \div 3 = 27$. We know that each row, column and diagonal has the same sum. Therefore the sum of each row = the sum of each column = the sum of each diagonal = 27.

We can now use this information to determine the values in each cell of the $3 \times 3$ grid. In the third column we know that $B + 13 + 3 = 27$ and $B = 11$ follows.

In the first row we know that $A + 1 + B = 27$ and $B = 11$ so $A + 1 + 11 = 27$. $A = 15$ follows.

In the second row we know that $5 + C + 13 = 27$ and $C = 9$ follows.

Then in the second column $1 + C + E = 27$ and $C = 9$ so $1 + 9 + E = 27$. $E = 17$ follows.

Since we know the values of $A$ and $E$ we can compute the sum $A + E = 15 + 17 = 32$.

Therefore, the sum $A + E$ is 32.
Solution 2

In the second solution we determine the required sum without finding the row/column/diagonal sum. Since the row sum equals the column sum we know that the sum of row 1 equals the sum of column 3.

\[ A + 1 + B = B + 13 + 3 \]
\[ A + 1 = 13 + 3 \quad \text{since} \quad B \text{ is common to both sides} \]
\[ \therefore A = 15 \]

Again, since the row sum equals the column sum we know that the sum of column 2 equals the sum of row 2.

\[ 1 + C + E = 5 + C + 13 \]
\[ 1 + E = 5 + 13 \quad \text{since} \quad C \text{ is common to both sides} \]
\[ \therefore E = 17 \]

Since we know the values of \( A \) and \( E \) we can compute the sum \( A + E = 15 + 17 = 32 \).

Therefore, the sum \( A + E \) is 32.

Solution 3

In solution 3 we find the sum \( A + E \) without finding any individual values.

\[
\text{Sum of Row 1} + \text{Sum of Column 2} = \text{Sum of Row 2} + \text{Sum of Column 3}
\]
\[ A + 1 + B + 1 + C + E = 5 + C + 13 + B + 13 + 3 \]

Since \( B + C \) is common to both sides, the equation simplifies to:

\[ A + 1 + 1 + E = 5 + 13 + 13 + 3 \]
\[ A + E + 2 = 34 \]
\[ \therefore A + E = 32 \]

Therefore, the sum \( A + E \) is 32.