



## Problem of the Week

### Problem D and Solution

#### Stay On Course

#### Problem

To describe the locations of towns in the State of Mathigan, Ima Hiker uses coordinates and coordinate geometry. Distances are measured in kilometres.

Ima starts her walk at Angletown which is located at  $A(0, 0)$ . She walks in a straight line to the northwest until she reaches Bisector Valley. Once she reaches Bisector Valley, she walks in a straight line to the northeast until she reaches Conic Centre which is located at  $C(0, 4)$ . She then walks in a straight line returning to Angletown. How far does Ima walk?

#### Solution

##### Solution 1

If you travel northwest from  $A(0, 0)$ , the line of travel will make a  $45^\circ$  angle to the negative  $x$ -axis and a  $45^\circ$  angle to the positive  $y$ -axis. Bisector Valley is located at  $B$ , somewhere on this line of travel. If you travel northeast from Bisector Valley to Conic Centre  $C(0, 4)$ , the line will intersect the  $y$ -axis at a  $45^\circ$  angle.

In  $\triangle ABC$ ,  $\angle BAC = \angle BCA = 45^\circ$ . It follows that  $\triangle ABC$  is isosceles and  $BC = BA = m$ , for some  $m > 0$ . Since two of the angles in  $\triangle ABC$  are  $45^\circ$ , then the third angle,  $\angle ABC = 90^\circ$  and the triangle is right-angled. The distance from Angletown to Conic Centre along the  $y$ -axis is  $AC = 4$  km.

Using the Pythagorean Theorem in  $\triangle ABC$ ,

$$\begin{aligned}AC^2 &= BC^2 + BA^2 \\4^2 &= m^2 + m^2 \\16 &= 2m^2 \\8 &= m^2 \\\sqrt{8} &= m, m > 0\end{aligned}$$

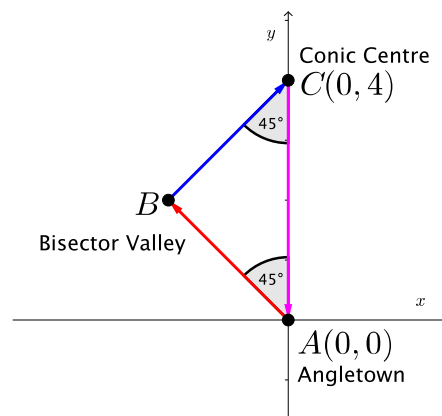
The total distance walked by Ima Hiker is  $BA + BC + AC = \sqrt{8} + \sqrt{8} + 4 = (2\sqrt{8} + 4)$  km.

Note that the answer  $(2\sqrt{8} + 4)$  is an *exact* answer. We can use a calculator to determine that this distance is approximately 9.7 km.

The exact total distance travelled can be further simplified as follows:

$$2\sqrt{8} + 4 = 2(\sqrt{4}\sqrt{2} + 4) = 2(2\sqrt{2} + 4) = 4\sqrt{2} + 4$$

This method of simplifying radicals is developed in later mathematics courses.



**Solution 2**

If you travel northwest from  $A(0,0)$ , the line of travel will make a  $45^\circ$  angle to the negative  $x$ -axis and a  $45^\circ$  angle to the positive  $y$ -axis. Bisector Valley is located at  $B$ , somewhere on this line of travel. From  $B$ , draw a line segment perpendicular to the  $y$ -axis, meeting the  $y$ -axis at  $D$ . A line of travel in a northeast direction from  $B$  to Conic Centre, at  $C$ , will make  $\angle DBC = 45^\circ$ .

In  $\triangle ABD$ ,  $\angle BAD = 45^\circ$  and  $\angle ADB = 90^\circ$ . It follows that  $\angle ABD = 45^\circ$ ,  $\triangle ABD$  is isosceles and  $BD = DA$ .

In  $\triangle CBD$ ,  $\angle CBD = 45^\circ$  and  $\angle CDB = 90^\circ$ . It follows that  $\angle BCD = 45^\circ$ ,  $\triangle CBD$  is isosceles and  $DC = BD$ .

The distance from Angletown,  $A(0,0)$ , to Conic Centre,  $C(0,4)$ , is 4 km along the positive  $y$ -axis. Since  $DC = DA$  and  $CA = DC + DA$ , then we know that  $DC = DA = 2$  km. But  $DC = BD$  so  $DC = BD = DA = 2$ . The known information is marked on the second diagram.

Using the Pythagorean Theorem in right-angled  $\triangle ABD$ ,

$$\begin{aligned} BA^2 &= BD^2 + DA^2 \\ BA^2 &= 2^2 + 2^2 \\ BA^2 &= 8 \\ BA &= \sqrt{8}, BA > 0 \end{aligned}$$

Using the same reasoning in  $\triangle CBD$ , we obtain  $BA = \sqrt{8}$ .

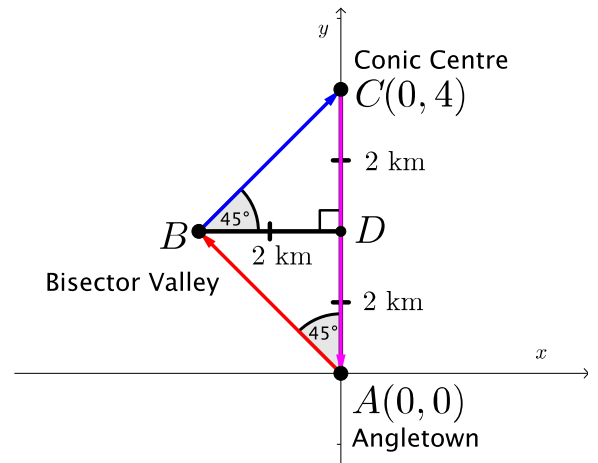
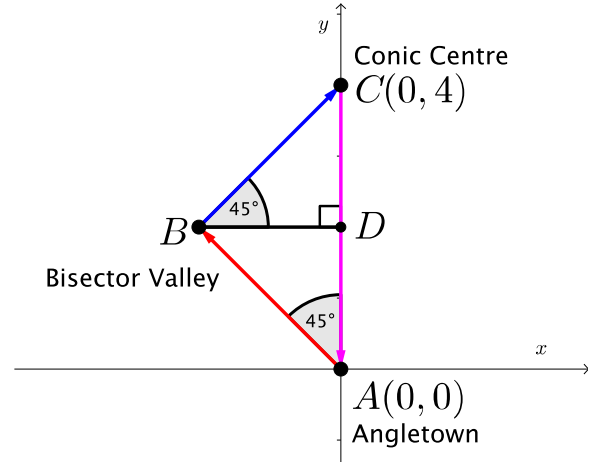
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**Solution 3**

If you travel northwest from  $A(0, 0)$ , the line of travel will make a  $45^\circ$  angle to the negative  $x$ -axis and a  $45^\circ$  angle to the positive  $y$ -axis. It follows that this line has slope  $-1$ . Since this line passes through  $A(0, 0)$  and has slope  $-1$ , the equation of the line through  $A$  and  $B$  is  $y = -x$ . (1)

Bisector Valley is located at  $B$ , somewhere on  $y = -x$ . A line drawn to the northeast would be perpendicular to a line drawn to the northwest. Since a line to the northwest has slope  $-1$ , it follows that a line to the northeast would have slope  $1$ . This second line passes through  $B$  and  $C$  so it has slope  $1$  and  $y$ -intercept  $4$ , the  $y$ -coordinate of  $C$ . The equation of the second line is  $y = x + 4$ . (2)

Since Bisector Valley is located on both (1) and (2), we can solve the system of equations to find the coordinates of  $B$ . In (1) and (2), since  $y=y$ ,

$$\begin{aligned} -x &= x + 4 \\ -2x &= 4 \\ x &= -2 \end{aligned}$$

Substituting  $x = -2$  into (1), we obtain  $y = 2$ . Bisector Valley is located at  $B(-2, 2)$ .

Using the distance formula, we can find the lengths of  $BA$  and  $BC$ .

$$BA = \sqrt{(-2 - 0)^2 + (2 - 0)^2} = \sqrt{4 + 4} = \sqrt{8}$$

$$BC = \sqrt{(0 + 2)^2 + (4 - 2)^2} = \sqrt{4 + 4} = \sqrt{8}$$

The distance from Angletown,  $A(0, 0)$ , to Conic Centre,  $C(0, 4)$ , is  $4$  km along the positive  $y$ -axis. That is,  $AC = 4$ .

The total distance walked by Ima Hiker is  $BA + BC + AC = \sqrt{8} + \sqrt{8} + 4 = (2\sqrt{8} + 4)$  km.

Note that the answer  $(2\sqrt{8} + 4)$  is an *exact* answer. We can use a calculator to determine that this distance is approximately  $9.7$  km.

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