Problem of the Week
Problem D and Solution
Medians and Altitudes

Problem

A median is a line segment drawn from the vertex of a triangle to the midpoint of the opposite side. An altitude is a line segment drawn from a vertex of a triangle to the opposite side intersecting at 90°.

In \( \triangle ABC \), \( \angle ACB = 90° \). An altitude is drawn in \( \triangle ABC \) from \( C \) to \( AB \) intersecting at \( D \). A median is drawn in \( \triangle ADC \) from \( A \) meeting \( DC \) at \( M \). The length of the median \( AM \) is 13 and \( AD \) is 12.

Determine the length of \( BD \).

Solution

Solution 1

In this solution we will use Pythagoras’ Theorem.

Since \( M \) is a median in \( \triangle ADC \), \( DM = MC = a \). Let \( AC = b \), \( BD = c \) and \( BC = d \). The variables and the given information, \( AD = 12 \) and \( AM = 13 \), are shown on the diagram.

Since \( \triangle ADM \) contains a right angle at \( D \), \( DM^2 = a^2 = AM^2 - AD^2 = 13^2 - 12^2 = 25 \) and \( a = 5 \) follows. Then \( DC = 2a = 10 \).

Since \( \triangle ADC \) contains a right angle at \( D \), \( AC^2 = b^2 = AD^2 + DC^2 = 12^2 + 10^2 = 244 \) and \( b = \sqrt{244} \) follows.

Since \( \triangle CDB \) contains a right angle at \( D \), \( CB^2 = BD^2 + DC^2 = c^2 + 10^2 = c^2 + 100 \).

\[ \therefore d^2 = c^2 + 100. \tag{1} \]

Since \( \triangle ABC \) contains a right angle at \( C \), \( AB^2 = AC^2 + BC^2 \). \( \therefore (12 + c)^2 = (\sqrt{244})^2 + d^2 \) which simplifies to \( 144 + 24c + c^2 = 244 + d^2 \).

This further simplifies to \( c^2 + 24c = 100 + d^2 \). \( \tag{2} \)

Substituting for \( d^2 \) from (1) into (2), we obtain \( c^2 + 24c = 100 + c^2 + 100 \). Simplifying we get \( 24c = 200 \) and \( c = \frac{25}{3} \) follows.

\[ \therefore \text{the length of } BD = \frac{25}{3}. \]
Solution 2

From the first solution, we will pick up the computed values $AC = \sqrt{244}$ and $DC = 10$.

In $\triangle ADC$ and $\triangle ACB$, $\angle ADC = \angle ACB = 90^\circ$ and $\angle DAC = \angle BAC$, a common angle. So $\triangle ADC \sim \triangle ACB$. From similar triangles, it follows that

$$\frac{AD}{AC} = \frac{AC}{AB} \quad \text{“Cross-Multiplying”,} \quad 12c + 144 = 244$$

$$12c = 100$$

$$c = \frac{25}{3}$$

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C
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But $c = BD$, so the length of $BD$ is $\frac{25}{3}$.

Solution 3

Position $\triangle ABC$ so that $AB$ lies along the y-axis, $C$ is on the positive x-axis and altitude $DC$ lies along the positive x-axis with $D$ at the origin. Since $AM$ is a median in $\triangle ADC$, $DM = MC = a$. Then $M$ has coordinates $(a,0)$ and $C$ has coordinates $(2a,0)$. Since $AD = 12$, $A$ has coordinates $(0,12)$. Since $B$ is on the y-axis, let $B$ have coordinates $(0,b)$ with $b < 0$.

In $\triangle ADM$, $DM^2 = AM^2 - AD^2 = 13^2 - 12^2 = 25$ and $DM = 5$ follows. Therefore $a = 5$, $2a = 10$ and $C$ has coordinates $(10,0)$.

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C(2a,0)
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slope $AC = \frac{12 - 0}{0 - 10} = \frac{-6}{5}$; slope $BC = \frac{b - 0}{0 - 10} = \frac{b}{-10}$

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C(10,0)
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Since $\angle ACB = 90^\circ$, $AC \perp BC$ and their slopes should be negative reciprocals.

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A(0,12)
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\[ \therefore \frac{b}{-10} = \frac{5}{6} \]

\[ 6b = -50 \]

\[ b = \frac{-25}{3} \]

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B(0,b)
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It then follows that the coordinates of $B$ are \(0, \frac{-25}{3}\). The length of $BD$ is the distance from $D$, the origin, to the point $B$ which is $\frac{25}{3}$ units.