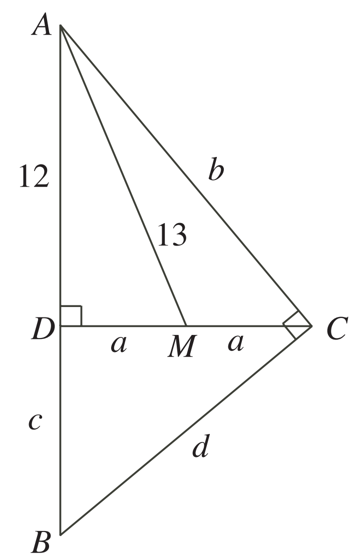
Problem of the Week
Problem D and Solution
Medians and Altitudes



Problem

A *median* is a line segment drawn from the vertex of a triangle to the midpoint of the opposite side. An altitude is a line segment drawn from a vertex of a triangle to the opposite side intersecting at 90°.

In $\triangle ABC$, $\angle ACB = 90^\circ$. An altitude is drawn in $\triangle ABC$ from C to AB intersecting at D. A median is drawn in $\triangle ADC$ from A meeting DC at M. The length of the median AM is 13 and AD is 12.

Determine the length of BD.

Solution
Solution 1

In this solution we will use Pythagoras' Theorem.

Since M is a median in $\triangle ADC$, $DM = MC = a$. Let $AC = b$, $BD = c$ and $BC = d$. The variables and the given information, $AD = 12$ and $AM = 13$, are shown on the diagram.

Since $\triangle ADM$ contains a right angle at D, $DM^2 = a^2 = AM^2 - AD^2 = 13^2 - 12^2 = 25$ and $a = 5$ follows. Then $DC = 2a = 10$.

Since $\triangle ADC$ contains a right angle at D, $AC^2 = b^2 = AD^2 + DC^2 = 12^2 + 10^2 = 244$ and $b = \sqrt{244}$ follows.

Since $\triangle CDB$ contains a right angle at D, $CB^2 = BD^2 + DC^2 = c^2 + 10^2 = c^2 + 100$.
$\therefore d^2 = c^2 + 100. \qquad (1)$

Since $\triangle ABC$ contains a right angle at C, $AB^2 = AC^2 + BC^2$. $\therefore (12 + c)^2 = (\sqrt{244})^2 + d^2$ which simplifies to $144 + 24c + c^2 = 244 + d^2$.

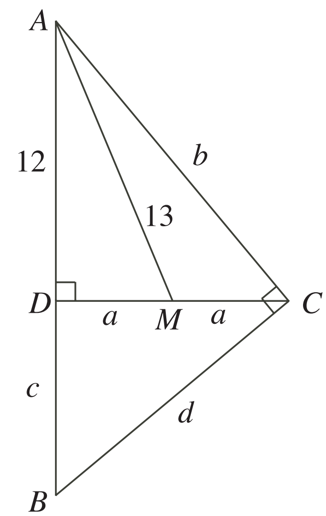
This further simplifies to $c^2 + 24c = 100 + d^2. \qquad (2)$

Substituting for d^2 from (1) into (2), we obtain $c^2 + 24c = 100 + c^2 + 100$. Simplifying we get $24c = 200$ and $c = \dfrac{25}{3}$ follows.

$\therefore$ the length of BD is $\dfrac{25}{3}$.

Solution 2

From the first solution, we will pick up the computed values $AC = \sqrt{244}$ and $DC = 10$.

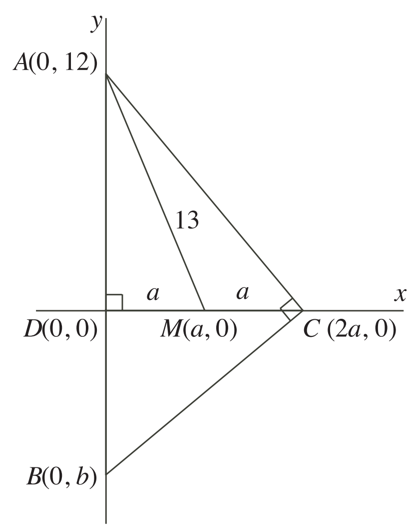


In $\triangle ADC$ and $\triangle ACB$, $\angle ADC = \angle ACB = 90^\circ$ and $\angle DAC = \angle BAC$, a common angle. So $\triangle ADC \sim \triangle ACB$. From similar triangles, it follows that

$$\begin{aligned} \frac{AD}{AC} &= \frac{AC}{AB} \\ \frac{12}{\sqrt{244}} &= \frac{\sqrt{244}}{12+c} \\ \text{“Cross-Multiplying”,} \qquad 12c + 144 &= 244 \\ 12c &= 100 \\ c &= \frac{25}{3} \end{aligned}$$

But $c = BD$, so the length of BD is $\frac{25}{3}$.

Solution 3



Position $\triangle ABC$ so that AB lies along the y-axis, C is on the positive x-axis and altitude DC lies along the positive x-axis with D at the origin. Since AM is a median in $\triangle ADC$, $DM = MC = a$. Then M has coordinates $(a, 0)$ and C has coordinates $(2a, 0)$. Since $AD = 12$, A has coordinates $(0, 12)$. Since B is on the y-axis, let B have coordinates $(0, b)$ with $b < 0$.

In $\triangle ADM$, $DM^2 = AM^2 - AD^2 = 13^2 - 12^2 = 25$ and $DM = 5$ follows. Therefore $a = 5$, $2a = 10$ and C has coordinates $(10, 0)$.

$$\text{slope } AC = \frac{12-0}{0-10} = \frac{-6}{5}; \;\; \text{slope } BC = \frac{b-0}{0-10} = \frac{b}{-10}$$

Since $\angle ACB = 90^\circ$, $AC \perp BC$ and their slopes should be negative reciprocals.

$$\begin{aligned} \therefore \frac{b}{-10} &= \frac{5}{6} \\ 6b &= -50 \\ b &= \frac{-25}{3} \end{aligned}$$

It then follows that the coordinates of B are $\left(0, \frac{-25}{3}\right)$. The length of BD is the distance from D, the origin, to the point B which is $\frac{25}{3}$ units.