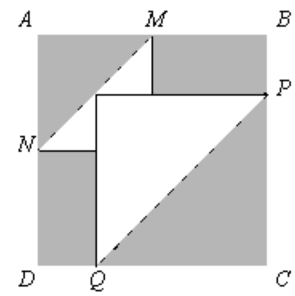
Problem of the Week

Problem D and Solution

Fold Carefully

Problem

A square piece of paper $ABCD$ has side length 20 cm. The page is grey on one side and white on the other. Point M is the midpoint of side AB and point N is the midpoint of side AD. The paper is folded along MN so that A touches the paper. Point C is then folded over a line PQ parallel to MN so that C lies on MN. What is the area of hexagon $NMBPQD$?

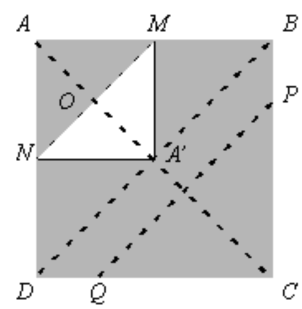


Solution

If we are able to find the area of $\triangle AMN$ and $\triangle PCQ$, we can subtract these areas from the area of square $ABCD$ to find the area of hexagon $NMBPQD$. To find the area of $\triangle PCQ$, we must find the length of PC and CQ.

Since M and N are midpoints of AB and AD respectively, $AM = \frac{1}{2}(AB) = 10$ and $AN = \frac{1}{2}(AD) = 10$. Therefore $AM = AN = 10$ and $\triangle AMN$ is an isosceles right triangle. It follows that $\angle ANM = \angle AMN = 45°$. Since PQ is parallel to MN, PQ also meets the sides of the square at a $45°$ angle. It follows that $\triangle PCQ$ is also an isosceles right triangle with $PC = CQ$ and $\angle CPQ = \angle CQP = 45°$.

After the first fold, let A touch the paper at A'. $\triangle MA'N$ is a reflection of $\triangle MAN$ in the line MN. It follows that $\angle AMN = \angle A'MN = 45°$ and $\angle ANM = \angle A'NM = 45°$. Then $\angle AMA' = \angle ANA' = 90°$. Since all four sides of $AMA'N$ are equal in length and all four corners are $90°$, $AMA'N$ is a square.



Since $\angle MAA' = \angle MAC = 45°$, the diagonal AA' of square $AMA'N$ lies along the diagonal AC of square $ABCD$. (In fact, A' lies at the intersection of the two diagonals of $ABCD$, the centre of the square.)

The length of the diagonal of square $AMA'N$ can be found using the Pythagorean Theorem.

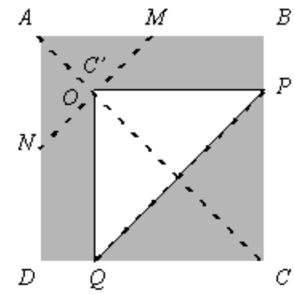
$$AA' = \sqrt{(AM)^2 + (MA')^2} = \sqrt{10^2 + 10^2} = \sqrt{200} = 10\sqrt{2}$$

Some students may not be familiar with simplifying radicals:

$$\sqrt{200} = \sqrt{100 \times 2} = \sqrt{100}\sqrt{2} = 10\sqrt{2}$$

Let O be the intersection of the two diagonals of square $AMA'N$. It is also the intersection of MN and AC. (We will show later that this is in fact the point of contact of C with the paper after the second fold.) Then $AO = \frac{1}{2}(AA') = \frac{1}{2}(10\sqrt{2}) = 5\sqrt{2}$.

We know that $\triangle PCQ$ is a right isosceles triangle. When the triangle is reflected in the line segment PQ, a square, $PCQC'$, is created with C' being the image of C. We will not present the argument here because it is very similar to the argument presented for $AMA'N$. Since $\angle PCC' = \angle PCA = 45°$, CC' lies along the diagonal CA. Also, C' is the intersection of CA with MN. This means that C' and O are the same point.



The length of the diagonal of square $ABCD$ can be calculated using the Pythagorean Theorem.

$$AC = \sqrt{(AB)^2 + (BC)^2} = \sqrt{20^2 + 20^2} = \sqrt{800} = 20\sqrt{2}$$

The length of CC' equals the length of AC subtract the length of OA.

$$CC' = 20\sqrt{2} - 5\sqrt{2} = 15\sqrt{2}$$

But $CC' = PQ$ so $PQ = 15\sqrt{2}$. Let $PC = CQ = x$. Then, using the Pythagorean Theorem, in $\triangle PCQ$,

$$\begin{aligned}
(PC)^2 + (CQ)^2 &= (PQ)^2 \\
x^2 + x^2 &= (15\sqrt{2})^2 \\
x^2 + x^2 &= 225 \times 2 \\
2x^2 &= 450 \\
x^2 &= 225 \\
x &= 15
\end{aligned}$$

We now have enough information to calculate the area of hexagon $NMBPQD$.

$$\begin{aligned}
\text{Area } NMBPQD &= \text{Area } ABCD - \text{Area } \triangle AMN - \text{Area } \triangle PCQ \\
&= AB \times BC - \frac{AM \times AN}{2} - \frac{PC \times CQ}{2} \\
&= 20 \times 20 - \frac{10 \times 10}{2} - \frac{15 \times 15}{2} \\
&= 400 - \frac{100}{2} - \frac{225}{2} \\
&= \frac{800}{2} - \frac{100}{2} - \frac{225}{2} \\
&= \frac{475}{2}
\end{aligned}$$

The area of hexagon $NMBPQD$ is $\frac{475}{2}$ cm^2 or 237.5 cm^2.