Problem of the Week
Problem D and Solution
Fold Carefully

Problem
A square piece of paper $ABCD$ has side length 20 cm. The page is grey on one side and white on the other. Point $M$ is the midpoint of side $AB$ and point $N$ is the midpoint of side $AD$. The paper is folded along $MN$ so that $A$ touches the paper. Point $C$ is then folded over a line $PQ$ parallel to $MN$ so that $C$ lies on $MN$. What is the area of hexagon $NMBPQD$?

Solution
If we are able to find the area of $\triangle AMN$ and $\triangle PCQ$, we can subtract these areas from the area of square $ABCD$ to find the area of hexagon $NMBPQD$. To find the area of $\triangle PCQ$, we must find the length of $PC$ and $CQ$.

Since $M$ and $N$ are midpoints of $AB$ and $AD$ respectively, $AM = \frac{1}{2}(AB) = 10$ and $AN = \frac{1}{2}(AD) = 10$. Therefore $AM = AN = 10$ and $\triangle AMN$ is an isosceles right triangle. It follows that $\angle ANM = \angle AMN = 45^\circ$. Since $PQ$ is parallel to $MN$, $PQ$ also meets the sides of the square at a $45^\circ$ angle. It follows that $\triangle PCQ$ is also an isosceles right triangle with $PC = CQ$ and $\angle CPQ = \angle CQP = 45^\circ$.

After the first fold, let $A$ touch the paper at $A'$. $\triangle MA'N$ is a reflection of $\triangle MAN$ in the line $MN$. It follows that $\angle AMN = \angle A'MN = 45^\circ$ and $\angle ANM = \angle A'NM = 45^\circ$. Then $\angle MA'N = \angle ANA' = 90^\circ$. Since all four sides of $\triangle MA'N$ are equal in length and all four corners are $90^\circ$, $\triangle MA'N$ is a square.

Since $\angle MA'A = \angle MAC = 45^\circ$, the diagonal $AA'$ of square $\triangle MA'N$ lies along the diagonal $AC$ of square $ABCD$. (In fact, $A'$ lies at the intersection of the two diagonals of $ABCD$, the centre of the square.)

The length of the diagonal of square $\triangle MA'N$ can be found using the Pythagorean Theorem.

$$AA' = \sqrt{(AM)^2 + (MA')^2} = \sqrt{10^2 + 10^2} = \sqrt{200} = 10\sqrt{2}$$

Some students may not be familiar with simplifying radicals:

$$\sqrt{200} = \sqrt{100 \times 2} = \sqrt{100} \sqrt{2} = 10\sqrt{2}$$

Let $O$ be the intersection of the two diagonals of square $\triangle MA'N$. It is also the intersection of $MN$ and $AC$. (We will show later that this is in fact the point of contact of $C$ with the paper after the second fold.) Then $AO = \frac{1}{2}(AA') = \frac{1}{2}(10\sqrt{2}) = 5\sqrt{2}$.
We know that $\triangle PCQ$ is a right isosceles triangle. When the triangle is reflected in the line segment $PQ$, a square, $PCQC'$, is created with $C'$ being the image of $C$. We will not present the argument here because it is very similar to the argument presented for $AMA'N$. Since $\angle PCC' = \angle PCA = 45^\circ$, $CC'$ lies along the diagonal $CA$. Also, $C'$ is the intersection of $CA$ with $MN$. This means that $C'$ and $O$ are the same point.

The length of the diagonal of square $ABCD$ can be calculated using the Pythagorean Theorem.

$$AC = \sqrt{(AB)^2 + (BC)^2} = \sqrt{20^2 + 20^2} = \sqrt{800} = 20\sqrt{2}$$

The length of $CC'$ equals the length of $AC$ subtract the length of $OA$.

$$CC' = 20\sqrt{2} - 5\sqrt{2} = 15\sqrt{2}$$

But $CC' = PQ$ so $PQ = 15\sqrt{2}$. Let $PC = CQ = x$. Then, using the Pythagorean Theorem, in $\triangle PCQ$,

$$(PC)^2 + (CQ)^2 = (PQ)^2$$

$$x^2 + x^2 = (15\sqrt{2})^2$$

$$2x^2 = 450$$

$$x^2 = 225$$

$$x = 15$$

We now have enough information to calculate the area of hexagon $NMBPQD$.

$$\text{Area } NMBPQD = \text{Area } ABCD - \text{Area } \triangle AMN - \text{Area } \triangle PCQ$$

$$= AB \times BC - \frac{AM \times AN}{2} - \frac{PC \times CQ}{2}$$

$$= 20 \times 20 - \frac{10 \times 10}{2} - \frac{15 \times 15}{2}$$

$$= 400 - \frac{100}{2} - \frac{225}{2}$$

$$= 800 - \frac{100}{2} - \frac{225}{2}$$

$$= \frac{475}{2}$$

The area of hexagon $NMBPQD$ is $\frac{475}{2}$ cm$^2$ or 237.5 cm$^2$. 
