Problem of the Week
Problem D and Solution
This Die is Magic

Problem

A die has the numbers 1, 2, 3, 4, 6, and 8 on its six faces. When this die is rolled, if an odd number appears on the top face, all of the odd numbers on the die magically double. For example, if the number appearing on the top face was a 1, the 1 and 3 on the original die would double and the other four numbers would remain the same. However, if an even number appears on the top face as a result of the roll, all of the even numbers on the die are halved. For example, if the number appearing on the top face was an 8, the 2, 4, 6, and 8 on the original die would change to half of their initial value and the other two numbers would remain the same. Suppose the die with 1, 2, 3, 4, 6, and 8 on its six faces is rolled once and changes as described above. The changed die is rolled again. No change occurs when the die is rolled this time. What is the probability that the number appearing on the top face after this roll is a 2?

Solution

Solution 1

In this solution, we will determine the possibilities for the first and second roll to count the total number of possible outcomes. We will then count the number of outcomes in which the second roll is a 2 and determine the probability.

If the first roll is odd, the numbers on the die change from $\{1, 2, 3, 4, 6, 8\}$ to $\{2, 2, 6, 4, 6, 8\}$ as a result of doubling the odd numbers. If we write the possible first and second rolls as an ordered pair, then the following 12 combinations are possible:

First roll 1: $(1,2), (1,2), (1,6), (1,4), (1,6), (1,8)$
First roll 3: $(3,2), (3,2), (3,6), (3,4), (3,6), (3,8)$

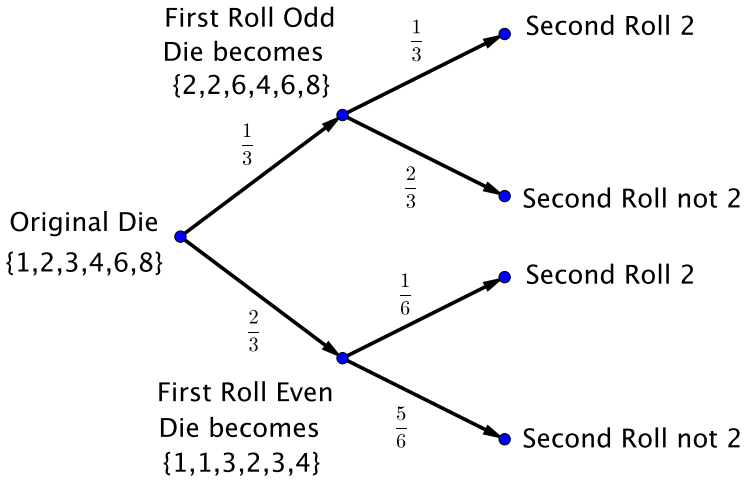
If the first roll is even, the numbers on the die change from $\{1, 2, 3, 4, 6, 8\}$ to $\{1, 1, 3, 2, 3, 4\}$ as a result of halving the even numbers. If we write the possible first and second rolls as an ordered pair, then the following 24 combinations are possible:

First roll 2: $(2,1), (2,1), (2,3), (2,2), (2,3), (2,4)$
First roll 4: $(4,1), (4,1), (4,3), (4,2), (4,3), (4,4)$
First roll 6: $(6,1), (6,1), (6,3), (6,2), (6,3), (6,4)$
First roll 8: $(8,1), (8,1), (8,3), (8,2), (8,3), (8,4)$

There are 36 total possible outcomes. Of these outcomes, 8 have a second roll of 2. The probability of rolling a 2 on the second roll is $\frac{8}{36} = \frac{2}{9}$.

Solution 2

In this solution, we will show the possibilities on a tree diagram.



The probability of rolling an odd number on the first roll is $\frac{2}{6} = \frac{1}{3}$.

The die then changes from $\{1, 2, 3, 4, 6, 8\}$ to $\{2, 2, 6, 4, 6, 8\}$ as a result of doubling the odd numbers. The probability of now rolling a 2 on the second roll is $\frac{2}{6} = \frac{1}{3}$. Multiplying the two probabilities, we obtain $\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$. This is the probability of rolling a 2 on the second roll after rolling an odd number on the first roll.

The probability of rolling an even number on the first roll is $\frac{4}{6} = \frac{2}{3}$.

The die then changes from $\{1, 2, 3, 4, 6, 8\}$ to $\{1, 1, 3, 2, 3, 4\}$ as a result of halving the even numbers. The probability of now rolling a 2 on the second roll is $\frac{1}{6}$. Multiplying the two probabilities, we obtain $\frac{2}{3} \times \frac{1}{6} = \frac{1}{9}$. This is the probability of rolling a 2 on the second roll after rolling an even number on the first roll.

Since the two cases are disjoint we add the probabilities to obtain $\frac{1}{9} + \frac{1}{9} = \frac{2}{9}$. The probability of rolling a 2 on the second roll is $\frac{2}{9}$.