



Problem of the Week

Problem C and Solution

Places for Pigeons

Problem

One hundred pigeons are to be housed in identical-sized cages under the following conditions:

- each cage must contain at least one pigeon;
- no two cages can contain the same number of pigeons; and
- no cages can go inside any other cage.

Determine the maximum number of cages required to house the pigeons.

Solution

In order to maximize the number of cages, each cage must contain the smallest number of birds possible. However, no two cages can contain the same number of pigeons. The simplest way to determine this number is to put one pigeon in the first cage and then let the number of pigeons in each cage after that be one more than the number of pigeons in the cage before it, until all 100 pigeons are housed.

Put 1 pigeon in the first cage, 2 pigeons in the second cage, 3 pigeons in the third cage, and so on. After filling 12 cages in this manner, we have

$1 + 2 + 3 + \cdots + 11 + 12 = 78$ pigeons housed. After putting 13 pigeons in the thirteenth cage, $78 + 13 = 91$ pigeons are housed. There are 9 pigeons left to house. But we already have a cage containing 9 pigeons. The remaining 9 pigeons must be distributed among the existing cages while maintaining the condition that no two cages contain the same number of pigeons.

The most obvious way to do this is to put the 9 pigeons in the last cage which already contains 13 pigeons. This would mean that the final cage would contain $13 + 9 = 22$ birds! A better solution might be to increase the number of birds in each of the final nine cages by one bird each. This solution is summarized below:

Cage #	1	2	3	4	5	6	7	8	9	10	11	12	13
# of birds	1	2	3	4	6	7	8	9	10	11	12	13	14

The maximum number of cages required is 13. If you had 14 cages, with the first cage holding 1 pigeon and each cage after that holding one more pigeon than the cage before, you could house 105 pigeons, five more than the number of pigeons that we have.

