Problem of the Week
Problem C and Solution
Sum Positive Primes

Problem

A *prime* number is a positive integer greater than 1 that has exactly two positive integer factors, 1 and the number itself. A *composite* number is a positive integer greater than 1 that has more than two positive integer factors. The number 1 is neither prime nor composite.

Four distinct prime numbers have a product of $d10$: a three-digit number with hundreds digit d. Determine all possible values of the sum of these four prime numbers.

Solution

Since the product $d10$ ends in 0, it must be divisible by 10, which is the product of the two primes 2 and 5.

When $d10$ is divided by 10, the quotient is $d10 \div 10 = d1$. This two-digit number must be composite and must be the product of two distinct prime numbers, neither of which is 2 or 5. We can rule out any two-digit prime numbers for $d1$ since these numbers would only have one prime factor. Therefore, we can rule out 11, 31, 41, 61, and 71: the five prime numbers ending with a 1. Then, d cannot be 1, 3, 4, 6, or 7. Since $d10$ is a three-digit number, $d \neq 0$ because $d10 = 010 = 10$ is not a three-digit number. The remaining possibilities for d are 2, 5, 8, and 9.

If $d = 2$, then the two-digit number would be 21, which has prime factors 7 and 3. The four prime factors of $d10 = 210$ are 2, 3, 5, and 7, producing a sum of $2 + 3 + 5 + 7 = 17$.

If $d = 5$, then the two-digit number would be 51, which has prime factors 17 and 3. The four prime factors of $d10 = 510$ are 2, 3, 5, and 17, producing a sum of $2 + 3 + 5 + 17 = 27$.

If $d = 8$, then the two-digit number would be 81, which is the product 9×9. There are already four factors, two of which are composite so $d10 = 810$ cannot be expressed as the product of four distinct prime numbers. Therefore, 8 can be ruled out as a possible value for d. (Note that $810 = 2 \times 3 \times 3 \times 3 \times 3 \times 5$, which is the product of six prime numbers not all of which are distinct.

If $d = 9$, then the two-digit number would be 91, which has prime factors 7 and 13. The four prime factors of $d10 = 910$ are 2, 5, 7, and 13, producing a sum of $2 + 5 + 7 + 13 = 27$. However, we already have the sum 27.

Since there are no other possible cases to consider, the only possible sums of the four distinct prime factors that multiply to $d10$ are 17 and 27.