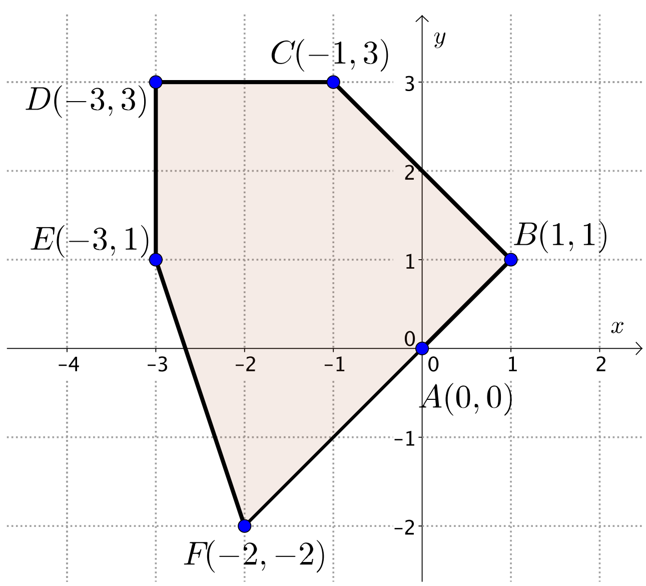
Problem of the Week
Problem C and Solution
Clean Up

Problem

Locations on the main floor of a house are described using a coordinate system and coordinate geometry. Vroom is a robotic vacuum that moves around the main floor of a house. The robot travels from one point to the next point in one straight line. Vroom's base is located at $(0,0)$. Vroom starts at its base and moves to the following points, in order, before returning to its base: $(1,1)$, $(-1,3)$, $(-3,3)$, $(-3,1)$, $(-2,-2)$. What is the area of the figure that Vroom traced out?

Solution

Our first task is to draw a diagram to represent Vroom's path. Label the points $A(0,0)$, $B(1,1)$, $C(-1,3)$, $D(-3,3)$, $E(-3,1)$, and $F(-2,-2)$.

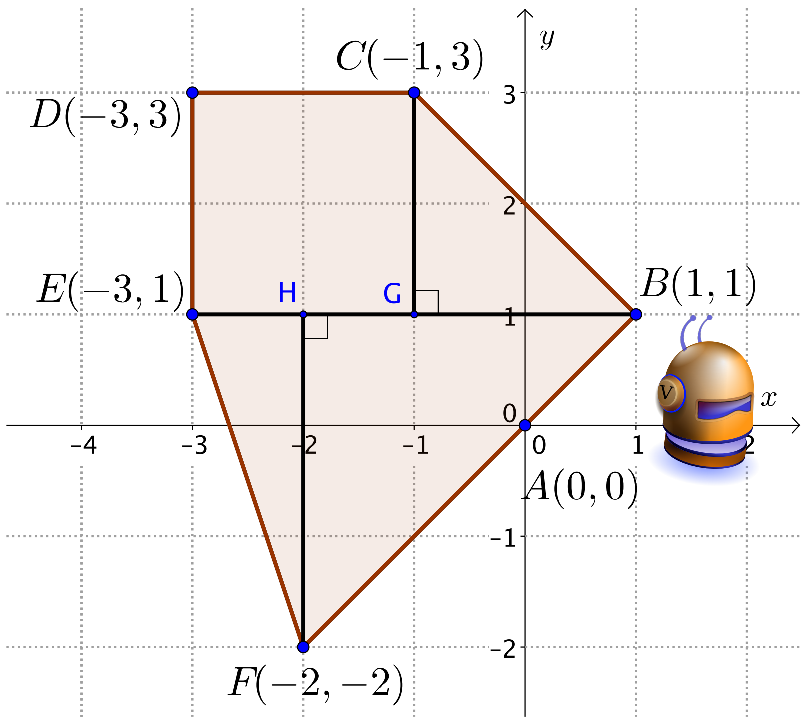


We will make a note here about points F, A and B. If we move 1 unit right and 1 unit up from $F(-2,-2)$, we get to $(-1,-1)$. If we move 1 unit right and 1 unit up from $(-1,-1)$, we get to $A(0,0)$. If we move 1 unit right and 1 unit up from $A(0,0)$, we get to $B(1,1)$. This illustrates that the points F, A and B lie on the same line. A detailed discussion of this is provided in high school math courses.

From here, we will present two possible solutions.

Solution 1: Divide the region into triangles and a square.

There are several ways to break the diagram into regions. On the diagram, we have square $CDEG$, triangle BCG, and triangle BEF. We will determine the area of each figure and then add them together to find the total area of the region that Vroom traced out.



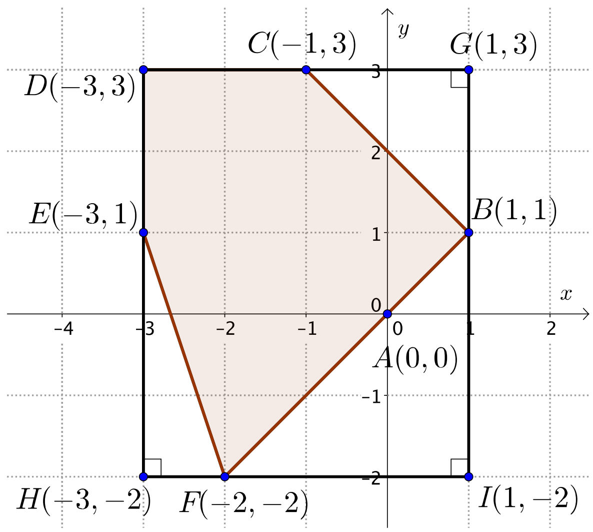
$$
\begin{aligned}
\text{area of square } CDEG &= CD \times DE = 2 \times 2 = 4 \text{ units}^2 \\
\text{area of triangle } BCG &= \frac{1}{2} \times GB \times CG = \frac{1}{2} \times 2 \times 2 = 2 \text{ units}^2 \\
\text{area of triangle } BEF &= \frac{1}{2} \times BE \times HF = \frac{1}{2} \times 4 \times 3 = 6 \text{ units}^2 \\
\text{Total area } ABCDEF &= 4 + 2 + 6 = 12 \text{ units}^2
\end{aligned}
$$

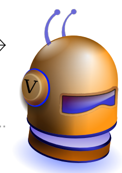
The total area of the region traced out by Vroom is 12 units^2.

If we had not used the information that F, A and B are three points on the same line we would have to further break up the region BEF into triangles and trapezoids to calculate the area.

Solution 2: Enclose the region with a rectangle.

On the diagram, we have drawn rectangle $DHIG$, the smallest rectangle that encloses the region $ABCDEF$. One of the horizontal sides passes through C and the other passes through F. One of the vertical sides passes through E and the other passes through B. The rectangle contains four shapes, the figure $ABCDEF$ traced out by Vroom, triangle CGB, triangle BIF and triangle EFH. To calculate the area of $ABCDEF$, we will subtract the areas of the three triangles from the area of the rectangle.





$$\begin{aligned}
\text{area of rectangle } DHIG &= DH \times DG = 5 \times 4 = 20 \text{ units}^2 \\
\text{area of triangle } CGB &= \frac{1}{2} \times CG \times GB = \frac{1}{2} \times 2 \times 2 = 2 \text{ units}^2 \\
\text{area of triangle } BIF &= \frac{1}{2} \times FI \times IB = \frac{1}{2} \times 3 \times 3 = 4.5 \text{ units}^2 \\
\text{area of triangle } EFH &= \frac{1}{2} \times HF \times HE = \frac{1}{2} \times 1 \times 3 = 1.5 \text{ units}^2 \\
\text{Total area } ABCDEF &= 20 - 2 - 4.5 - 1.5 = 12 \text{ units}^2
\end{aligned}$$

The total area of the region traced out by Vroom is 12 units^2.

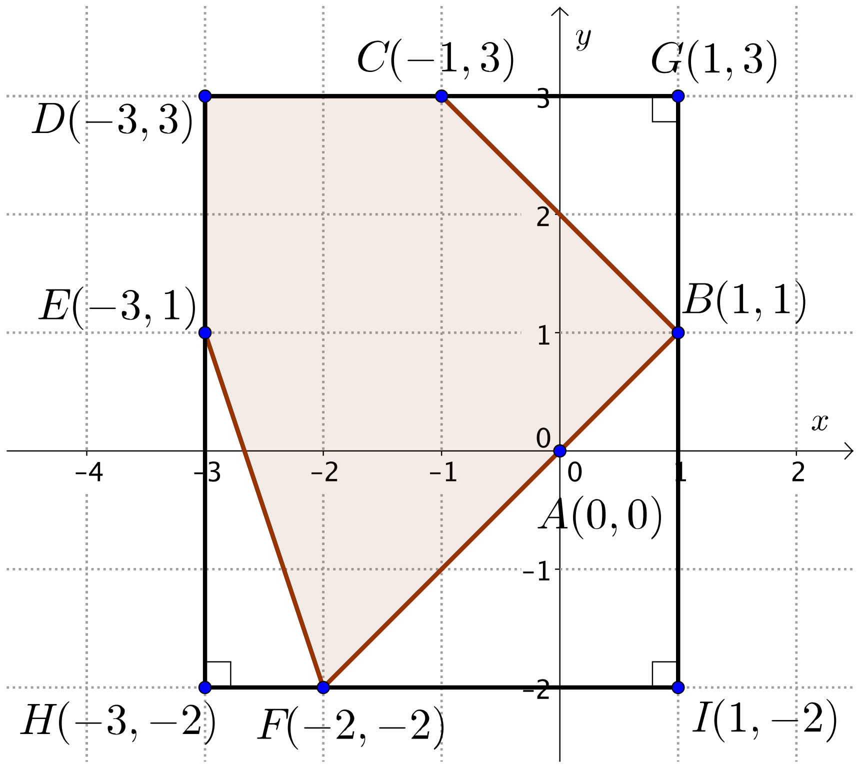
If we had not used the information that F, A and B are three points on the same line we would have to break triangle BIF into a triangle and a trapezoid or two triangles and a rectangle to calculate the area.

Extension:

What is the total distance travelled by Vroom? (The full solution is on the next page.)

Solution to the Extension

We will use the diagram from the second solution to help in determining the lengths of the sides of $ABCDEF$. To find the lengths of the sides of the right-angle triangles we will use the Pythagorean Theorem.



In triangle CGB, $BC^2 = CG^2 + GB^2 = 2^2 + 2^2 = 4 + 4 = 8$ and $BC = \sqrt{8}$.
In triangle BIF, $FB^2 = BI^2 + IF^2 = 3^2 + 3^2 = 9 + 9 = 18$ and $FB = \sqrt{18}$.
In triangle EFH, $EF^2 = EH^2 + HF^2 = 3^2 + 1^2 = 9 + 1 = 10$ and $EF = \sqrt{10}$.

The length of $CD = 2$ and the length of $DE = 2$.

The total distance travelled by Vroom is the sum of all of the side lengths of $ABCDEF$.

$BC + CD + DE + EF + FB = \sqrt{8} + 2 + 2 + \sqrt{10} + \sqrt{18} = 4 + \sqrt{8} + \sqrt{10} + \sqrt{18}$

The total distance travelled by Vroom is $(4 + \sqrt{8} + \sqrt{10} + \sqrt{18})$ units. Using a calculator, this distance is approximately 14.2 units.