Problem of the Week
Problem C and Solution
Around We Go

Problem
A circle with centre $O$ has a point $A$ on the circumference. Radius $OA$ is rotated $20^\circ$ clockwise about the centre, resulting in the image $OB$. Point $A$ is then connected to point $B$. Radius $OB$ is rotated $20^\circ$ clockwise about the centre, resulting in the image $OC$. Point $B$ is then connected to point $C$.

The process of clockwise rotations continues until some radius rotates back onto $OA$. Every point on the circumference is connected to the points immediately adjacent to it as a result of the process. A polygon is created.

a) Determine the number of sides of the polygon.

b) Determine the sum of the angles in the polygon. That is, determine the sum of the angles at each of the vertices of the polygon.

Solution
Each time the process is repeated, another congruent triangle is created. Each of these triangles has a $20^\circ$ angle at $O$, the centre of the circle. But a complete rotation at the centre is $360^\circ$. Since each angle in the triangles at the centre of the circle is $20^\circ$ and the total measure at the centre is $360^\circ$, then there are $360 \div 20 = 18$ triangles formed. This means that there are 18 distinct points on the circumference of the circle and the polygon has 18 sides. An 18-sided polygon is called an octadecagon, from octa meaning 8 and deca meaning 10.

The other two angles in the each of the congruent triangles are equal. (Two sides of the triangle are radii of the circle. The triangles are therefore isosceles.) The angles in a triangle sum to $180^\circ$ so after the $20^\circ$ angle is removed, there is $160^\circ$ remaining for the other two angles. It follows that each of the other two angles in each triangle measures $160^\circ \div 2 = 80^\circ$. The following diagram illustrates this information for the two adjacent triangles $AOB$ and $BOC$.

Each angle in the polygon is formed by an $80^\circ$ angle from one triangle and the adjacent $80^\circ$ angle from the next triangle. For example, $\angle ABC = \angle ABO + \angle OBC = 80^\circ + 80^\circ = 160^\circ$.

There are 18 vertices in the octadecagon and the angle at each vertex is $160^\circ$. Therefore the sum of the angles in the octadecagon is $18 \times 160^\circ = 2880^\circ$.

Diagrams are provided on the next page to further support the solution.
Angle at the Centre of a Circle

Completing the Construction

The Octadecagon - 18 sided Polygon

Notice the vertices of the octadecagon are labelled $A$ to $S$, but the letter $O$ is missing since it was used in the original construction as the centre of the circle.