Problem of the Week
Problem C and Solution
Probably Even

Problem

Four distinct integers are to be chosen from the integers 1, 2, 3, 4, 5, 6, and 7. How many different selections are possible so the sum of the four integers is even?

Solution

We could look at every possible selection of four distinct numbers from the list, determine the sum of each selection and then count the number of selections for which the sum is even. There are 35 different selections to examine. A justification of this number is provided on the second page of this solution. This would not be an efficient approach!

We will make two simple observations. First, when even numbers are added together the sum is always even. And second, in order to produce an even sum using odd numbers, an even number of odd numbers is required in the sum. We will use these observations to break the problem into cases in which the sum is even. There are three cases to consider.

1. **No Odd Numbers are Selected**

   Since there are only three even numbers, namely 2, 4, and 6, it is not possible to select only even numbers. Therefore, there are no selections in which there are no odd numbers.

2. **Exactly Two Odd Numbers are Selected**

   There are four choices for the first odd number. For each of these four choices, there are three choices for the second number producing $4 \times 3 = 12$ choices for two odd numbers. However, each choice is counted twice. For example, 1 could be selected first and 3 could be chosen second or 3 could be selected first and 1 could be chosen second. Therefore, there are only $12 \div 2 = 6$ selections of two odd numbers. They are $\{1, 3\}$, $\{1, 5\}$, $\{1, 7\}$, $\{3, 5\}$, $\{3, 7\}$, and $\{5, 7\}$. For each of the 6 possible selections of two distinct odd numbers, we need to select two even numbers from the three even numbers in the list. We could use a similar argument to the selection of the two odd numbers or simply list the (three) possibilities: $\{2, 4\}$, $\{2, 6\}$, and $\{4, 6\}$. Therefore, there are $6 \times 3 = 18$ selections of four distinct numbers in which exactly two of the numbers are odd.

3. **Exactly Four Odd Numbers are Selected**

   Since there are only four odd numbers in the list to choose from, there is only one way to select four distinct odd numbers from the list.

We have considered every possible case in which the selection produces an even sum. Therefore, there are $0 + 18 + 1 = 19$ selections of four distinct numbers from the list such that the sum is even.

Why are there 35 ways to select four different numbers from the list?

In the solution on the previous page we counted the selections in which the sum was even. There were 19 possibilities. The remaining selections must produce an odd sum. There are two possibilities: either there is 1 odd number and 3 even numbers, or there are 3 odd numbers and 1 even number.

If there is 1 odd number and 3 even numbers, there are only four possible selections, namely, $\{1, 2, 4, 6\}$, $\{3, 2, 4, 6\}$, $\{5, 2, 4, 6\}$, and $\{7, 2, 4, 6\}$. Once the odd number is selected, the 3 even numbers, $\{2, 4, 6\}$, must be selected.

If there are 3 odd numbers and 1 even number, there are twelve possible selections. The 3 odd numbers can be selected in four ways, namely, $\{1, 3, 5\}$, $\{1, 3, 7\}$, $\{1, 5, 7\}$, and $\{3, 5, 7\}$. For each of these 4 selections of three odd numbers, the even number can be selected in 3 ways producing $4 \times 3 = 12$ possible selections of four distinct numbers in which three of the numbers are odd and the other is even.

We have considered all possible ways in which four distinct numbers can be selected from the list. The total number of selections is $19 + 4 + 12 = 35$.

We can arrive at this number in a different way.

There are 7 choices for the first number. For each of these choices for the first number, there are 6 choices for the second number, or $7 \times 6 = 42$ choices for the first two numbers. For each of these 42 choices for the first two numbers, there are 5 choices for the third number, or $42 \times 5 = 210$ choices for the first three numbers. For each of these 210 choices of the first three numbers, there are 4 choices for the final number, or $210 \times 4 = 840$ selections of the four numbers. This is considerably higher than the 35 choices shown above!

Our 840 selections assume that the order of selection is important. Each selection has been counted 24 times. To justify this, we will look at the number of ways a specific four number selection can be arranged. Without loss of generality, we will consider the selection $\{1, 2, 3, 4\}$. The 1 could be placed in four spots. For each of these four placements of the 1, the 2 could be placed in three spots producing $4 \times 3 = 12$ ways of placing the 1 and 2. For each of these twelve placements of the 1 and 2, the 3 could be placed in two spots producing $12 \times 2 = 24$ ways of placing the 1, 2 and 3. Once the numbers 1, 2, and 3 are placed, the 4 must be placed in the remaining spot. There are 24 ways of arranging the four numbers. We have to divide 840 by 24 since we have counted each selection 24 times.

Therefore, there are $840 \div 24 = 35$ ways to select four different numbers from the list of seven numbers.