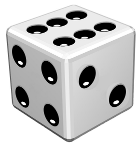

Problem of the Week
Problem C and Solution
That's Not Fair Or Is It?

Problem

One die has the even numbers 2, 4, 6, 8, 10, and 12 on its faces and the other die has the odd numbers 1, 3, 5, 7, 9, and 11 on its faces. A turn consists of rolling the dice and using the two numbers that appear on the top faces. Anna and Elle take turns rolling the dice.

Anna performs the following steps after each roll to determine whether or not she gets a point. First, Anna determines the sum, S, of the numbers on the top faces. On the roll shown above, $S = 9$. Then, using S, Anna determines, D, the digit sum. If S is a single digit number, then D is the same as S. If S is a two digit number, then D is the sum of the two digits. (If the roll is a 6 and a 3 like above, then the digit sum and the sum are both 9. If the roll is a 5 and 10, then the sum is 15 and the digit sum is $1 + 5 = 6$. If the roll is a 9 and 10, then the sum is 19 and the digit sum is $1 + 9 = 10$.) Anna gets a point if the digit sum D is a multiple of 4. Elle gets a point if one of the numbers on the top face is a multiple of the number on the other top face. With the dice roll shown above, Elle would get a point since 6 is a multiple of 3. Is this game fair? That is, do Anna and Elle have the same probability of getting a point on any roll? Justify your answer.

Solution

First, determine the number of possible rolls. For the even numbered die, there are six possible numbers that could appear on the top face. For each of these six possibilities, there are six possible numbers that could appear on the top face of the odd die. There are a total of $6 \times 6 = 36$ possible rolls.

For Anna to get a point she must have a roll that produces a digit sum that is a multiple of 4.

Table of Roll Sums

		Even Die Roll					
		2	4	6	8	10	12
Odd Die Roll	1	3	5	7	9	11	13
	3	5	7	9	11	13	15
	5	7	9	11	13	15	17
	7	9	11	13	15	17	19
	9	11	13	15	17	19	21
	11	13	15	17	19	21	23

The next table calculates the digit sums from the roll sums. Remember that the digit sum and roll sum are the same for single digit sums.

Table of Digit Sums

		Even Die Roll					
		2	4	6	8	10	12
Odd Die Roll	1	3	5	7	9	$1+1=2$	$1+3=4$
	3	5	7	9	$1+1=2$	$1+3=4$	$1+5=6$
	5	7	9	$1+1=2$	$1+3=4$	$1+5=6$	$1+7=8$
	7	9	$1+1=2$	$1+3=4$	$1+5=6$	$1+7=8$	$1+9=10$
	9	$1+1=2$	$1+3=4$	$1+5=6$	$1+7=8$	$1+9=10$	$2+1=3$
	11	$1+3=4$	$1+5=6$	$1+7=8$	$1+9=10$	$2+1=3$	$2+3=5$

If the digit sum is a multiple of 4, then Anna gets a point. From the table we see that there are 2 digit sums which are a multiple of 4. These digit sums are 4 and 8. The digit sum 4 occurs six times in the above table and the digit sum 8 occurs four times in the above table. This totals ten possible outcomes for Anna and her probability of scoring a point on any roll is $\frac{10}{36}$.

Elle has far less work to do to determine when she gets a point. None of the odd numbers are multiples of the even numbers. All multiples of even numbers are even and hence will never be odd.

Whenever a 1 is rolled on the die containing only odd numbers, Elle will score a point. That is, each of the 6 even numbers is a multiple of 1.

When a 3 is rolled on the die containing only odd numbers, Elle will score a point if the number on the top face of the even die is a 6 or 12. That is, only 2 of the even numbers are multiples of 3.

When a 5 is rolled on the die containing only odd numbers, Elle will score a point if the number on the top face of the even die is a 10. That is, only 1 of the even numbers is a multiple of 5.

None of the numbers on the die containing only even numbers is a multiple of 7, 9 or 11.

So Elle will score a point on $6+2+1=9$ of the 36 possible rolls. Therefore, Elle's probability of scoring a point on any roll is $\frac{9}{36}$.

The game is not fair since Anna's probability of scoring a point on any roll is greater than Elle's probability of scoring a point on any roll. It should be noted that the probabilities are close to being the same so one might say that this game is almost fair.