Problem of the Week
Problem B
Fenced In!

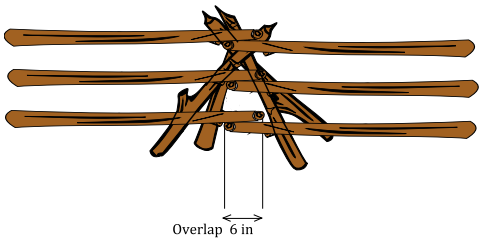
Problem

Cassie is building a cedar split-rail fence around her horse paddock. The three rails are laid horizontally, with the top ones supported in the notch of each four-post bundle, and the other two suspended by wires above them. Where they meet, each pair of rails overlap somewhat to provide stability for the fence.

a) Suppose the 3 rails in Cassie's fence have a 6 inch ($\frac{1}{2}$ ft) overlap, as shown in the diagram below. If she needs a fence 105 ft long, and each of her cedar rails is 10 ft long, how many rails will she need in total for the horizontal spans of her fence?

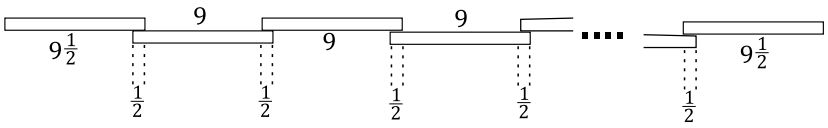
b) Cassie also needs to construct the four-post bundles for her fence. If the posts are to be 5 ft high in each bundle, how many cedar rails will she need for the bundles?

c) Suppose instead that Cassie can get cedar rails 13 ft long. How many of these rails would she need in total to construct her 105 ft fence?



Solution

a) First we need to think about how many spans of 3 rails Cassie needs. Since each rail is 10 ft long, and the fence is to be 105 ft long, she will need at least 11 spans. But we must also take into account the loss of $\frac{1}{2}$ ft for each internal overlap. Here is a diagram to illustrate what one level of horizontal rails would look like.



Thus, if we look at having 11 spans, there would be 2 end rails of $9\frac{1}{2}$ ft, and 9 internal spans of 9 ft, plus 10 overlaps of $\frac{1}{2}$ ft each. So 11 spans would give a total length of

$$2 \times 9\frac{1}{2} + 9 \times 9 + 10 \times \frac{1}{2} = 19 + 81 + 5 = 105 \text{ ft},$$

exactly the length of fence Cassie needs. Thus she would need $11 \times 3 = 33$ cedar rails in total for the horizontal spans of her fence.

b) The 11 horizontal spans will require 12 four-post bundles, since she needs them at both ends of the fence. Thus she will need $12 \times 4 = 48$ posts, each 5 ft high. Since each cedar rail is 10 ft long, she can get 2 posts from each rail, so she will need 24 cedar rails for the bundles. In total, Cassie would need $33 + 24 = 57$ of the 10 ft long cedar rails.

c) Since $105 \div 13 \approx 8.1$, we will need more than 8 spans of the 13 ft horizontal rails. Let's try using 9 spans; replacing 9 by 12 in our calculation from part a), we see that 9 spans would give a total length of

$$2 \times 12\frac{1}{2} + 7 \times 12 + 8 \times \frac{1}{2} = 25 + 84 + 4 = 113 \text{ ft}.$$

Thus 9 spans are sufficient. Since we exceeded the 105 ft by 8 ft the final span would only need to be 5 ft long. Cassie would only need two of the 13 ft rails to get the three horizontal 5 ft pieces for that end. So for the 9 horizontal spans, she would need $8 \times 3 + 2 = 26$ of the 13 ft cedar rails.

In addition, she would need 10 four-post bundles, each of which would require 2 cedar rails, since she could only cut 2 posts 5 ft high from each 13 ft cedar rail. Thus she would need 20 cedar rails for the bundles, giving a total of 46 rails for the entire fence.