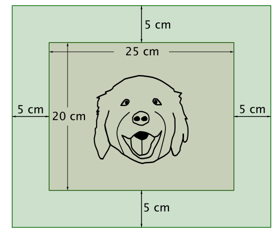
Problem of the Week

Problem B

Leo's Dog was Framed!

Problem

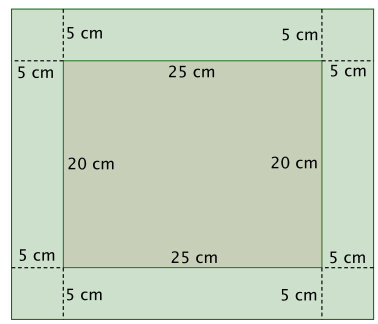
Leo built a picture frame using the plan illustrated to the right. The picture inside the frame is 25 cm wide and 20 cm high. The horizontal and vertical distance from the edge of the picture to the outer edge of the frame is 5 cm. This information is marked on the plan. What is the area of the border of the frame? There are many ways to solve this problem. Try to solve the problem in more than one way.



Extension: The area of the border of the frame is larger than the area of the picture inside the frame. The width of the current border is 5 cm. What would the width of the border need to be so that the area of the border equals the area of the picture? Determine your answer correctly rounded to one decimal place.

Solution

Solution 1

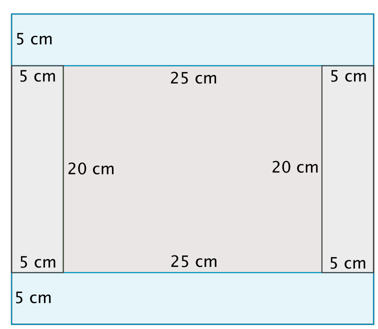


Draw vertical and horizontal lines along the edge of the picture extending to the outside of the frame. This creates four 5 cm by 5 cm squares (in the corners), two 25 cm by 5 cm rectangles (along the top and bottom), and two 5 cm by 20 cm rectangles (along the left and right side).

$$\begin{aligned}\text{Area of Border} &= \text{Area of four squares} + \text{Area of Two Rectangles} + \text{Area of Two Rectangles}\\ &= 4 \times 5 \times 5 + 2 \times 25 \times 5 + 2 \times 5 \times 20\\ &= 100 + 250 + 200\\ &= 550 \text{ cm}^2\end{aligned}$$

The area of the border of the frame is 550 cm^2.

Solution 2



Draw horizontal lines along the top and bottom edges of the picture extending to the outside of the frame. This creates two identical rectangles (top and bottom), and two other identical rectangles (left and right side of the frame). The length of the rectangles across the top and bottom is $5 + 25 + 5 = 35$ cm and the width is 5 cm. The two side rectangles have length 20 cm and width 5 cm.

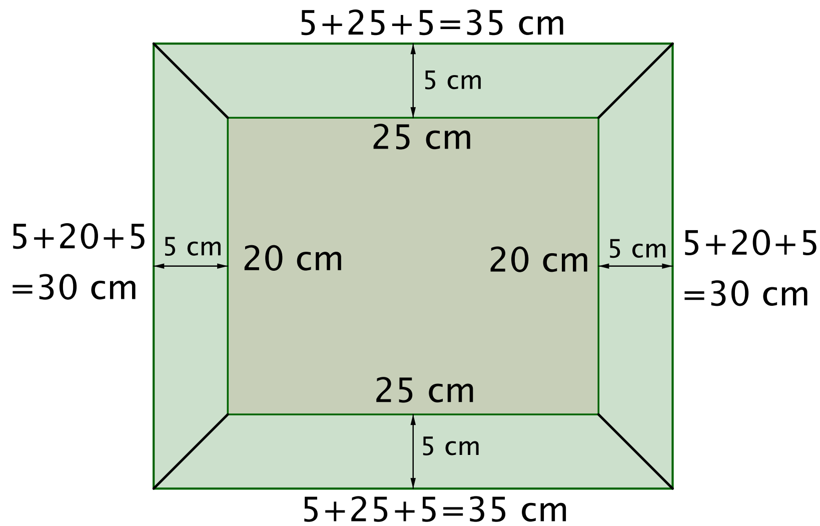
$$\begin{aligned}\text{Area of Border} &= \text{Area of top and bottom rectangles} + \text{Area of the two side rectangles}\\ &= 2 \times 35 \times 5 + 2 \times 20 \times 5\\ &= 350 + 200\\ &= 550 \text{ cm}^2\end{aligned}$$

The area of the border of the frame is 550 cm^2.

Solution 3

A trapezoid is a quadrilateral with two parallel sides. To find the area of a trapezoid, we could break it into two triangles and a rectangle. There is a formula for the area of a trapezoid. To find the area, find the sum of the lengths of the two parallel sides, a and b. Multiply this sum by the distance between the two parallel sides, h. Divide the product by 2. This can be written $A = h \times (a + b) \div 2$.

Draw diagonal lines from the four corners of the picture to the closest corner of the frame. This creates two identical trapezoids (top and bottom), and two other identical trapezoids (along the sides of the frame). The length of the parallel sides of the trapezoids across the top and bottom is $5 + 25 + 5 = 35$ cm and 25 cm. The length of the parallel sides of the trapezoids along the two sides is $5 + 20 + 5 = 30$ cm and 20 cm. The height of all four trapezoids is 5 cm.



$$\begin{aligned}
\text{Area of Border} &= \text{Area of top and bottom trapezoids} + \text{Area of the two side trapezoids}\\
&= 2 \times 5 \times (35 + 25) \div 2 + 2 \times 5 \times (30 + 20) \div 2\\
&= 300 + 250\\
&= 550 \text{ cm}^2
\end{aligned}$$

The area of the border of the frame is 550 cm^2.

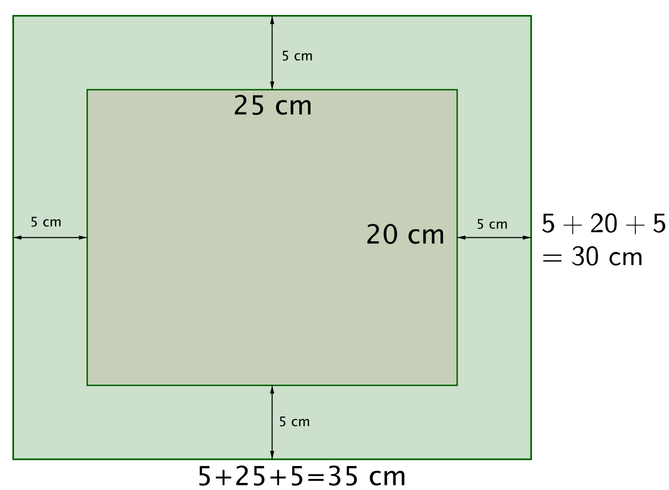
Solution 4

In this solution, we find the area of the larger outside rectangle and then subtract the area of the small interior rectangle. This difference is the area of the border.

The outside rectangle (the entire frame) has length $5 + 25 + 5 = 35$ cm and width $5 + 20 + 5 = 30$ cm. The area of the outer rectangle is $35 \times 30 = 1050$ cm^2.

The inside rectangle (the picture) has length 25 cm and width 20 cm. The area of the outer rectangle is $25 \times 20 = 500$ cm^2.

By subtracting the area of the inside rectangle from area of the large rectangle we determine the border area.



Therefore, the area of the border of the frame is

$$1050 - 500 = 550 \text{ cm}^2.$$

Solution to the extension

For the extension we will use the idea from solution 4. We know that the area of the border is 550 cm^2 when the width of the border is 5 cm. The area of the picture is 500 cm^2. We know that the width of the border must be less than 5 cm. We will basically try educated guesses to arrive at the width of the border correct to one decimal.

Try a border width of 4.5 cm.

Width of Border (in cm)	Length of Outer Rectangle (in cm)	Width of Outer Rectangle (in cm)	Area of Outer Rectangle) (in cm)^2)	Area of Inner Rectangle (in cm^2)	Area of Border (in cm^2)
4.5	$4.5 + 25 + 4.5$ $= 34$	$4.5 + 20 + 4.5$ $= 29$	34×29 $= 986$	500	$986 - 500$ $= 486$

Since the area of the border is under 500 cm^2, the width of the border is too small.

Try a border width of 4.7 cm.

Width of Border (in cm)	Length of Outer Rectangle (in cm)	Width of Outer Rectangle (in cm)	Area of Outer Rectangle) (in cm)^2)	Area of Inner Rectangle (in cm^2)	Area of Border (in cm^2)
4.7	$4.7 + 25 + 4.7$ $= 34.4$	$4.7 + 20 + 4.7$ $= 29.4$	34.4×29.4 $= 1011.36$	500	$1011.36 - 500$ $= 511.36$

Since the area of the border is over 500 cm^2, the width of the border is too large.

The only possible one decimal border width left to check is 4.6 cm.

Width of Border (in cm)	Length of Outer Rectangle (in cm)	Width of Outer Rectangle (in cm)	Area of Outer Rectangle) (in cm)^2)	Area of Inner Rectangle (in cm^2)	Area of Border (in cm^2)
4.6	$4.6 + 25 + 4.6$ $= 34.2$	$4.6 + 20 + 4.6$ $= 29.2$	34.2×29.2 $= 998.64$	500	$998.64 - 500$ $= 498.64$

Since the area of the border is under 500 cm^2, the width of the border is too small.

However, we want the width correct to one decimal so that the area of the border is as close to 500 cm^2 as possible. When the width is 4.6 cm, the area of the border is 498.64 cm^2, 1.36 cm^2 under the required area. When the width is 4.7 cm, the area of the border is 511.36 cm^2, 11.36 cm^2 over the required area.

Therefore, when the width of the border is 4.6 cm, the area of the border and the area of the picture are as close to being equal as possible under the condition of one decimal accuracy.

This method is not very efficient. There is good news. You will learn an algebraic approach to this problem in later math studies that will get you a value for the width much more efficiently. You will have to wait.