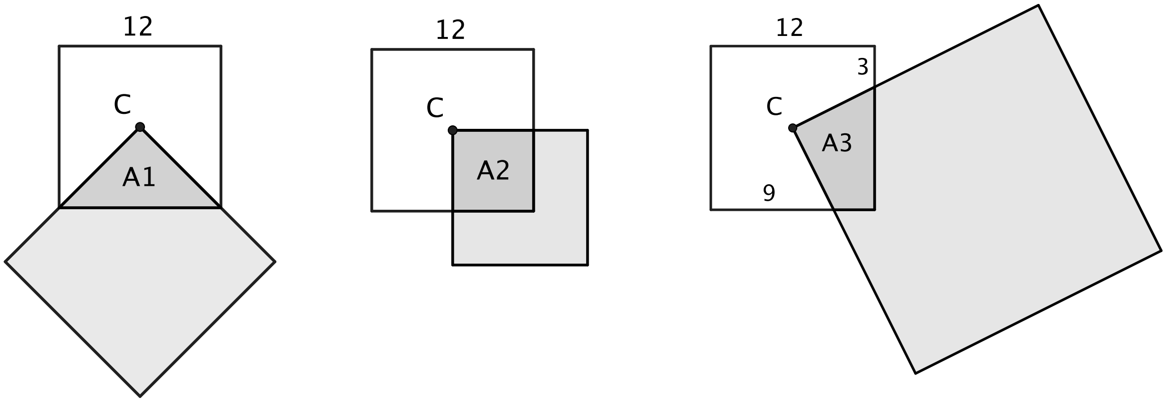
Problem of the Week
Problem B
Looks Can Be Deceiving

Problem

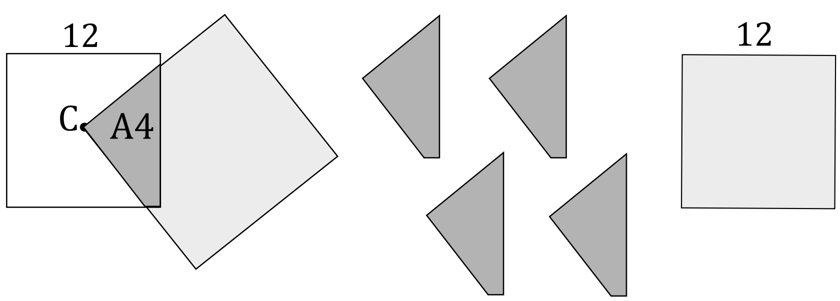
Using grid paper, you may wish to construct each of the diagrams shown below. This may be helpful as you work through the problem.

In each of the three diagrams below, there are two squares which overlap in various ways. The bottom square is fixed and has side length 12. The overlapping square has one vertex located at the centre C of the bottom square. The darker shaded regions are common to the two squares and are labeled $A1$, $A2$ and $A3$, respectively.



a) Determine the areas of $A1$, $A2$ and $A3$, the overlapping areas in each of the above diagrams. How is each of these areas related to the total area of the square of side length 12?

b) Below is a diagram in which the overlapping area, $A4$, is in an arbitrary position. Show that the blank square of side length 12 can be completely tiled by four copies of $A4$ and use this to determine the overlapping area, $A4$.



Extension: The above example shows us that the overlapping area of the two squares is 36 units^2, one-quarter of the total area. Would it make any difference if the overlapping figure were a rectangle? A right-angled triangle? Another polygon? Explain your thinking.

Solution

The solution is on the next page.

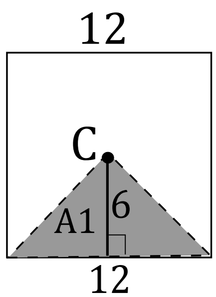
a) $A1$ is a triangle of height 6 and base 12, and thus has area $\frac{1}{2} \times 12 \times 6 = 36$ square units.

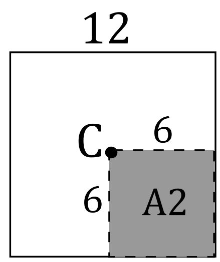
$A2$ is a square of side 6, with area $6 \times 6 = 36$ square units.

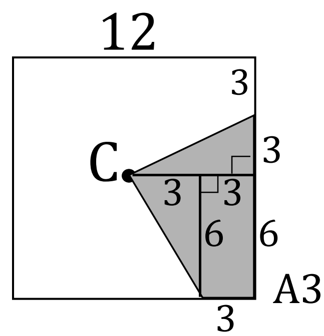
$A3$ consists of two right-angled triangles of base length 3 and height 6, plus a rectangle with side lengths 3 and 6.

The area of each right-angled triangle is $\frac{1}{2} \times 3 \times 6 = 9$ square units.
The area of the rectangle is $3 \times 6 = 18$ square units.
The total area of $A3$ is $9 + 9 + 18 = 36$ square units.

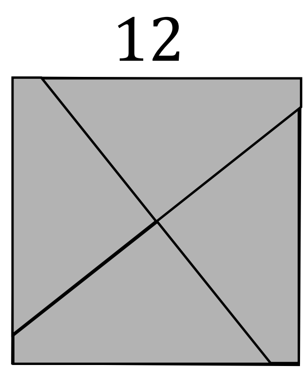
So all three overlapping areas $A1$, $A2$, and $A3$ have exactly the same area!







B) The way in which $A4$ tiles the square of side 12 is illustrated at the right. Since this shows that the overlapping area is 36 square units (i.e., $\frac{1}{4}$ of the area of the bottom square) for any position in which the overlapping figure is an irregular quadrilateral, and part a) shows that this also holds for the two special cases, we see that the overlapping area is always the same, 36 square units or $\frac{1}{4}$ of the area of the square with side length 12.



Extension: As long as the overlapping polygon has one vertex which is a right angle pinned at the centre, C, of the base square and the two sides forming that vertex are long enough that they intersect the sides of the bottom square, the shape of the overlapping area would always be one of type $A1$, $A2$, or $A4$, with area equal to $\frac{1}{4}$ of the area of the bottom square.