Problem of the Week

Problem B

Who Was Pythagoras?

Problem

Part of the side length of the large square shown in Figure 1 is *5* units. The remainder of the side length is *12* units. It follows that the total side length of the square in Figure 1 is $5 + 12 = 17$ units.

Within the large square in Figure 1, there are four right-angled triangles, which we shall call $A1$, $A2$, $A3$ and $A4$, and two squares $S1$ and $S2$.

a) How do you know that all four triangles are congruent to one another?

b) What are the areas of $S1$ and $S2$?

c) Figure 2 contains the same large square of side length 17 as in Figure 1. Figure 2 contains five geometric shapes. What must be the shape of the interior figure S? Explain your answer, stating the length of the sides of S, and the area of S.

d) Now compare Figure 1 to Figure 2. Which areas are the same in both Figure 1 and Figure 2? What areas of Figure 1 must sum to give the area of S?

e) Write your result from d) as an equation. This will tell you how the side lengths **5, 12, 13** of a right-angled triangle are related.

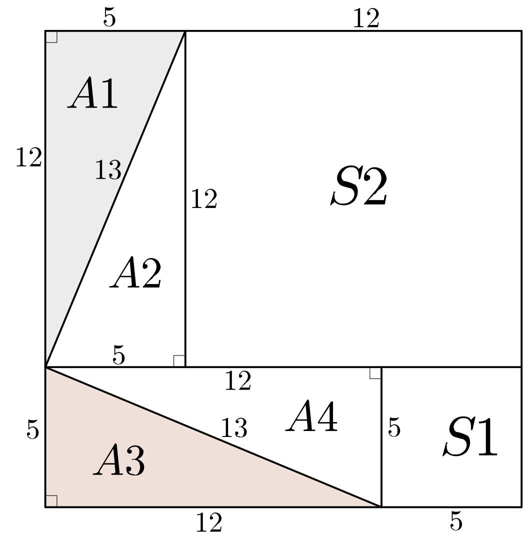

Figure 1

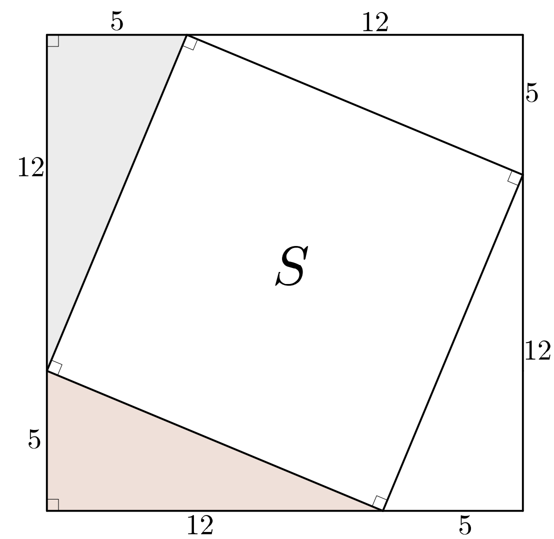

Figure 2

Solution

a) The four triangles are congruent to one another because each one is exactly half of a rectangle with sides 5 and 12, created by a diagonal of length 13. (Alternatively, we know they are congruent because they all have three sides of the same lengths.)

b) Since $S1$ and $S2$ are squares of sides 5 and 12 respectively, they have areas $5 \times 5 = 25$ and $12 \times 12 = 144$ (sometimes 5×5 is written as 5^2 and 12×12 is written as 12^2).

c) Since the four corner triangles are identical, their third side must be the same, with length 13 in each case (from Figure 1). Further, since all four corners of S are right angles, S is a square with side length 13 and area $13 \times 13 = 169$ (sometimes 13×13 is written as 13^2).

d) The four corner triangles of Figure 2 have the same areas as triangles $A1$, $A2$, $A3$, $A4$ in Figure 1. Since the large outer square in both figures has side length 17, the total area for both must be the same. Thus the area of S in Figure 2 must equal the sum of the areas of $S1$ and $S2$ in Figure 1.

e) Expressed as an equation, the result of part d) states that

$$\begin{aligned} \text{Area S1} + \text{Area S2} &= \text{Area S} \\ 25 \quad + \quad 144 &= \quad 169 \\ 5 \times 5 + 12 \times 12 &= 13 \times 13 \end{aligned}$$

This can be written $$5^2 \quad + \quad 12^2 \quad = \quad 13^2$$

This result is a specific example illustrating the famous Pythagorean Theorem. In a right-angled triangle the side opposite the right angle is called the hypotenuse. If the length of the hypotenuse is c and the lengths of the two shorter sides are a and b, then the Pythagorean Theorem relates the three sides with $a^2 + b^2 = c^2$.

If you do this problem again using a instead of 5, b instead of 12, and c instead of 13, then through your discovery you will actually prove the Pythagorean Theorem! This generalization is shown on the next page.

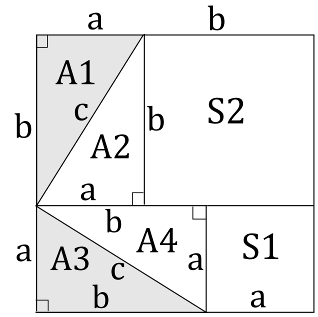

Figure 1

a) How do you know that all four triangles are congruent to one another?

The four triangles are congruent to one another because each one is exactly half of a rectangle with sides a and b, created by a diagonal of length c. (Alternatively, we know they are congruent because they all have three sides of the same lengths.)

b) What are the areas of $S1$ and $S2$?

Since $S1$ and $S2$ are squares of sides a and b respectively, they have areas $a \times a$ and $b \times b$ (sometimes written as a^2 and b^2).

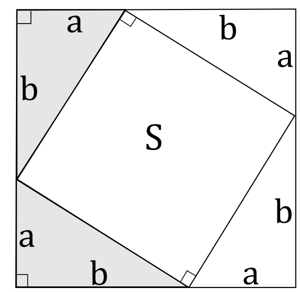

Figure 2

c) Figure 2 contains the same large square of side length c as in Figure 1. Figure 2 contains five geometric shapes. What must be the shape of the interior figure S? Explain your answer, stating the length of the sides of S, and the area of S.

Since the four corner triangles are identical, their third side must be the same, with length c in each case (from Figure 1). Further, the symmetry of the diagram implies that all four of the corner angles of S must be right angles. Thus S is a square with side length c and area $c \times c$, or c^2.

d) Now compare Figure 1 to Figure 2. Which areas are the same in both Figure 1 and Figure 2? What areas of Figure 1 must sum to give the area of S?

The four corner triangles of Figure 2 have the same areas as triangles $A1$, $A2$, $A3$, $A4$ in Figure 1. Since the large outer square in both figures has side $a + b$, the total area for both must be the same. Thus the area of S in Figure 2 must equal the sum of the areas of $S1$ and $S2$ in Figure 1.

e) Write your result from d) as an equation. This will tell you how the side lengths **a, b, c** of a right-angled triangle are related.

Expressed as an equation, the result of part d) states that

$$\text{Area S1} + \text{Area S2} = \text{Area S}$$

$$a \times a \quad + \quad b \times b \quad = \quad c \times c$$

This can be written

$$a^2 \quad + \quad b^2 \quad = \quad c^2$$

This result illustrates the famous Pythagorean Theorem which relates the side lengths a and b of a right-angled triangle to the hypotenuse, its longest side, length c. And through this discovery, the result has actually been proven!