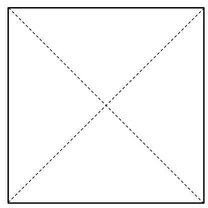
Problem of the Week
Problem A and Solution
Origami

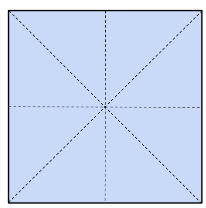
Problem

Laila likes to do origami which is the art of paper folding. Just by folding paper in a particular way, she can make all sorts of different animals. Many of the animals start with the same steps.

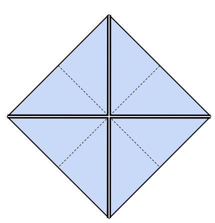
Laila starts with a square piece of paper. She folds it in half to form a triangle. Then she opens up the paper to start with a square again. She folds it in the opposite direction and also forms a triangle. When she opens it up again she can see creases on the paper that look like this:



Then she turns the paper over and folds it in half to form a rectangle. She opens up the paper and folds it in the opposite direction to form another rectangle. When she opens up the paper this time, she sees creases in the paper that look like this:



The centre of the square is the point where all of the creases intersect. Now, she takes each corner of the square and folds the paper so that each corner touches the centre of the square. Folding all four corners in this way forms another square.



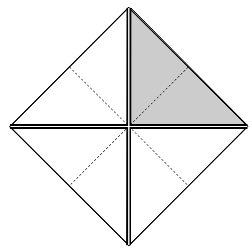
What fraction of the area of the original square is the area of the smaller square? Justify your answer.

Solution

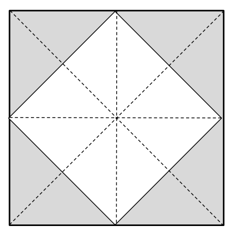
The smaller square has an area that is $\frac{1}{2}$ the area of the original square. There are several ways in which you can justify this.

Solution 1:

One way is to look at the following picture. Notice that the shaded triangle covers an area underneath it that is exactly the same size. That is true for all four of the triangles that have their points meet at the middle.



If we opened up the paper again, we could make the following observations:

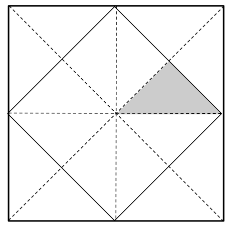


- The shaded parts of the original square each have a matching unshaded part.
- The shaded parts make up the area of the smaller square.
- This means that the area of the original square is 2 times the area of the smaller square.

It follows that the area of the smaller square is $\frac{1}{2}$ the area of the original square.

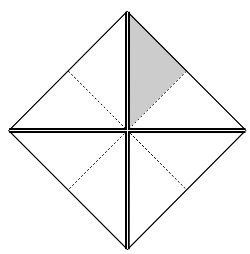
Solution 2:

Another way to show that the area of the smaller square is $\frac{1}{2}$ the area of the original square is to count the number of small triangles that are formed by the creases in the paper. One such triangle is shaded in the following diagram.



Each of these triangles have the same area. One way of showing that they all have the same area would be to cut up the square into the triangles and stack them on top of each other. If you count the number of small triangles in the original square there are 16 of them.

If you count the number of small triangles in the smaller square, there are 8 of them.



Since the smaller square is formed by half the number of small triangles as compared to the original square, then the smaller square has $\frac{1}{2}$ the area of the original square.

Teacher's Notes

Origami can be used to demonstrate many different geometric shapes, especially triangles and quadrilaterals. The instructions in this problem start with a square and then make folds to form a triangle, a rectangle, and then a smaller square.

There are many books and websites that can show you the steps required to create simple and complex origami figures. If you follow the steps to creating complex figures, you may see several different polygons, including a parallelogram, a trapezoid, and a *kite*. A kite is a quadrilateral with adjacent sides that are the same length.

Both the triangle formed in the first diagonal fold, and the smaller triangles formed by the creases are all right-angled, isoceles triangles. It is relatively easy to confirm that the triangle from the first diagonal fold satisfies this condition. Since you start with a square, and the interior angles of a square are all 90 degrees, then the corner of the triangle where the points of the square meet must be 90 degrees. Therefore the triangle is a right-angled triangle. Since the other two sides of the triangle are sides of the original square, those sides must be equal. Therefore it is an isoceles triangle. Also, we can determine the size of the other two interior angles of this triangle with logical deduction. When you fold the square on the diagonal and then open it up again, the crease that you see has bisected the angle in the corner of the square. Since that angle is 90 degrees, then the angle formed by the side of the square and the crease is half that size, which is 45 degrees. It would take a longer argument to show that the 16 smaller triangles are also right-angled, isoceles triangles. You could convince yourself it is true by cutting up the square into the 16 pieces and comparing the sides of a pair of triangles. You could also check that one corner of one of the triangles aligns with the corner of a square or a rectangle to confirm that its angle is 90 degrees.

Being able to make logical deductions in geometry is important. Finding the correct answer to a problem is not the only thing mathematicians and computer scientists care about. They are also concerned with describing the process for finding the correct answer and **proving** that the answer is correct. Problems in geometry can be a good place to practice these skills.