



The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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March 2017 Solutions

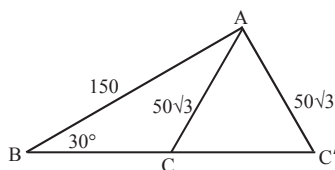
In honour of the 50th anniversary of the Faculty of Mathematics, at the beginning of each month of 2017, a set of five problems from the 54 years of CEMC contests will be posted. Solutions to the problems will be posted at the beginning of the next month. Hopefully, these problems will intrigue and inspire your mathematical mind. For more problem solving resources, please visit cemc.uwaterloo.ca.

1. 1976 Euclid Contest, Question B1

Triangle ABC has $\angle B = 30^\circ$, $AB = 150$, and $AC = 50\sqrt{3}$. Determine the length of BC .

Solution

Two triangles are possible, ABC and ABC' as depicted below.



By the cosine law,

$$\begin{aligned}AC^2 &= AB^2 + BC^2 - 2(AB)(BC) \cos(\angle B) \\(50\sqrt{3})^2 &= 150^2 + BC^2 - 2(150)(BC) \cos(30^\circ) \\7500 &= 22500 + BC^2 - 300(BC)\left(\frac{\sqrt{3}}{2}\right) \\0 &= BC^2 - 150\sqrt{3}BC + 15000 \\0 &= (BC - 50\sqrt{3})(BC - 100\sqrt{3})\end{aligned}$$

Therefore, $BC = 50\sqrt{3}$ or $BC = 100\sqrt{3}$.

2. 1965 Junior Mathematics Contest, Question 28

A man walks from A to B at 4 km/h and from B to C at 3 km/h. Then he walks from C to B at 6 km/h. and from B to A at 4 km/h. If the total time taken for the walk is 6 hours and $AB \neq BC$, then the total distance walked, in kilometers, is

- (A) 24 (B) 17 (C) 12 (D) $\frac{17}{2}$ (E) none of these

Solution

Let the distance from A to B be x km and the distance from B to C be y km.

Walking at 4 km/h from A to B takes $\frac{x}{4}$ hours.

Walking at 3 km/h from B to C takes $\frac{y}{3}$ hours.

Similarly, the return trip from C to B to A takes $\frac{y}{6} + \frac{x}{4}$ hours.
Since the total journey takes 6 hours, then

$$\begin{aligned}\frac{x}{4} + \frac{y}{3} + \frac{y}{6} + \frac{x}{4} &= 6 \\ 3x + 4y + 2y + 3x &= 72 \\ 6x + 6y &= 72 \\ 2x + 2y &= 24\end{aligned}$$

In total, the man walks $x + y + y + x = 2x + 2y$ km, which is 24 km.

ANSWER: (A)

3. 1971 Junior Mathematics Contest, Question 25

In the expression $S = \sqrt{x_1 - x_2 + x_3 - x_4}$, the variables x_1, x_2, x_3, x_4 , are replaced by 1, 2, 3, 4, with no repetitions allowed. If there are 24 possible replacements, then the number of times S will be a real number is

- (A) 8 (B) 12 (C) 13 (D) 14 (E) 16

Solution

The expression $S = \sqrt{x_1 - x_2 + x_3 - x_4}$ is a real number exactly when $x_1 - x_2 + x_3 - x_4 \geq 0$.

We re-write this inequality as $(x_1 + x_3) - (x_2 + x_4) \geq 0$.

If $(x_1, x_2, x_3, x_4) = (1, 2, 3, 4)$, then $(x_1 + x_3) - (x_2 + x_4) = -2 < 0$.

We note that $(1, 2, 3, 4)$, $(3, 2, 1, 4)$, $(1, 4, 3, 2)$, $(3, 4, 1, 2)$ give the same value. (In these quadruples, x_1 and x_3 are interchanged, as are x_2 and x_4 .) None of these four give a real number value for S .

If $(x_1, x_2, x_3, x_4) = (1, 3, 2, 4)$, then $(x_1 + x_3) - (x_2 + x_4) = -4 < 0$.

This quadruple represents 4 arrangements, namely $(1, 3, 2, 4)$, $(2, 3, 1, 4)$, $(1, 4, 2, 3)$, $(2, 4, 1, 3)$.

If $(x_1, x_2, x_3, x_4) = (1, 2, 4, 3)$, then $(x_1 + x_3) - (x_2 + x_4) = 0 \geq 0$.

This quadruple represents 4 arrangements.

If $(x_1, x_2, x_3, x_4) = (2, 1, 3, 4)$, then $(x_1 + x_3) - (x_2 + x_4) = 0 \geq 0$.

This quadruple represents 4 arrangements.

If $(x_1, x_2, x_3, x_4) = (2, 1, 4, 3)$, then $(x_1 + x_3) - (x_2 + x_4) = 2 \geq 0$.

This quadruple represents 4 arrangements.

If $(x_1, x_2, x_3, x_4) = (3, 1, 4, 2)$, then $(x_1 + x_3) - (x_2 + x_4) = 4 \geq 0$.

This quadruple represents 4 arrangements.

In total, 16 of the arrangements give a value for $(x_1 + x_3) - (x_2 + x_4)$ that is at least 0, and so a real value for S .

ANSWER: (E)

4. 1964 Junior Mathematics Contest, Question 32

Among grandfather's papers an old bill was found: "72 turkeys for \$_679_". The first and last digits of the number that represented the total price of the turkeys are replaced here with blanks as they have faded and are now illegible. The sum of the missing digits is

- (A) 6 (B) 8 (C) 7 (D) 5 (E) 3

Solution

Let the total price of 72 turkeys be $A679B$ cents.

Since each turkey has the same price, then $A679B$ must be divisible by 72.

An integer is divisible by 72 exactly when it is divisible by 8 and by 9.

An integer is divisible by 8 exactly when the three-digit integer formed by its last three digits is divisible by 8.

In other words, $A679B$ is divisible by 8 exactly when $79B$ is divisible by 8.

The only multiple of 8 in the range 790 to 799, inclusive, is 792, so $B = 2$.

Therefore, the price is $A6792$ cents.

An integer is divisible by 9 exactly when the sum of its digits is divisible by 9.

Here, $A + 6 + 7 + 9 + 2 = A + 24$. Since A is a digit between 1 and 9, then for $A + 24$ to be a multiple of 9, $A = 3$.

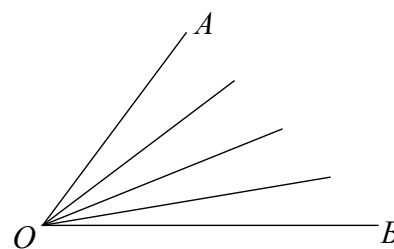
Therefore, the price is \$367.92 (which is divisible by 72) and so the sum of the missing digits is $3 + 2 = 5$.

ANSWER: (D)

5. 1979 Junior Mathematics Contest, Question 3

The number of acute angles in the diagram, where AOB is acute, is

- (A) 6 (B) 8 (C) 10
(D) 12 (E) 16



Solution

There are five rays from O .

Choose one ray for one angle arm (five choices).

Next, choose another ray for the second angle arm (four choices).

This gives $5 \times 4 = 20$ acute angles.

But each angle has been counted twice (eg. $\angle AOB$ and $\angle BOA$ have been counted separately).

Therefore, there are $20 \div 2 = 10$ acute angles.

ANSWER: (C)