Grade 6 Math Circles

March 9, 2022

The A, B, Γ's of Math - Problem Set

Exercise Solutions

Exercise A

Try to spell your name using Greek letters.

Exercise A Solution

There is no correct answer, unless you have a native Greek name. There are only 24 letters in the Greek Alphabet, so not every English letter can be accounted for.
My name is Daniel, so I might use Delta, alpha, nu, iota, epsilon and then lambda? I get $\Delta\alpha\nu\iota\epsilon\lambda$, which probably sounds nothing like my name if you pronounce the Greek letters as intended.

Exercise B

Determine what the variables are in each mathematical equation. If you are familiar with the equation, determine what each variable represents.

(α) $C = 2\pi r$ (circumference of a circle)

(β) $a^2 + b^2 = c^2$ (Pythagorean Theorem)

Exercise B Solution

(α) C is the variable for the circumference and r is the variable for the radius of a circle. Note that π is not a variable but a **constant** ($\pi \approx 3.14159$).

(β) a, b, and c are variables for the side lengths of a right-angled triangle and c specifically is the hypotenuse.

Exercise Γ

We say that the constant τ (**tau**) has the following relationship to π.

$$\tau = 2\pi$$

(α) Find the approximate value of τ to 2 decimal places.

(β) Find an approximate circumference of a circle with radius $r = 7$. Round to 2 decimal places.

(γ) Find the exact value of the circumference of a circle with radius $r = 28$.

Exercise Γ Solution

(α) $\tau \approx 6.28$

(β) Using π we have $C = 2\pi r$, but we can instead use $C = \tau r$. We find $C = 2\pi(7) = \tau(7) \approx (6.28) \times 7 = 43.69$.

(γ) $C = 28\tau$ (but $C = 56\pi$ works too!)

Exercise Δ

Amogh rolls a dice 9 times and records the result. Calculate the mean, μ, of his data:

$$\{2, 4, 6, 5, 5, 3, 2, 4, 5\}$$

Recall that we calculate the mean by taking the sum of the numbers in the list and dividing by how many numbers there are.

Exercise Δ Solution

There are 9 numbers in the data set. We have

$$\begin{aligned}\mu &= \frac{2+4+6+5+5+3+2+4+5}{9}\\ \mu &= \frac{36}{9}\\ \mu &= 4\end{aligned}$$

Exercise E

Given that $f(x) = 3x$ and $g(x) = x + 6$ calculate the following.

(α) $f(5)$

(β) $g(9)$

(γ) $g(104) - f(20)$

Exercise E Solution

(α) $f(5) = 3(5) = 15$

(β) $g(9) = (9) + 6 = 15$

(γ) $g(104) - f(20) = [(104) + 6] - [3(20)] = 110 - 60 = 50$

Exercise Z

Calculate the following.

(α) $\sigma(32)$

(β) $\sigma(37)$

Exercise Z Solution

(α) $\sigma(32) = 1 + 2 + 4 + 8 + 16 + 32 = 63$

(β) $\sigma(37) = 1 + 37 = 38$

Exercise H

What is $\pi(19)$? What about $\pi(22)$?

Predict the value of $\pi(100)$ given that $\pi(90) = 24$.

Exercise H Solution

$$\pi(19) = \pi(20) = \pi(21) = \pi(22) = 8$$

This is because between 19 and 22, there are no "new" prime numbers to count.

Since there is exactly 1 prime number between 90 and 100 (namely 97), we predict that

$$\pi(100) = \pi(90) + 1 = 24 + 1 = 25$$

Exercise I

Evaluate each sum.

(α) $\displaystyle\sum_{n=0}^{7} n$

(β) $\displaystyle\sum_{i=18}^{21} i$

Exercise I Solution

(α) $\displaystyle\sum_{n=0}^{7} n = 0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$

(β) $\displaystyle\sum_{i=18}^{21} i = 18 + 19 + 20 + 21 = 78$

Exercise Θ

Write out the expanded form of each summation and then evaluate the sum.

(α) $\sum_{n=1}^{10}(2n-1)$

(β) $\sum_{n=1}^{4}\left(\frac{1}{n+2}\right)$

Exercise Θ Solution

(α) This is the sum of the first 10 odd numbers.

$$\begin{aligned}\sum_{n=1}^{10}(2n-1) &= (2(1)-1)+(2(2)-1)+(2(3)-1)+\cdots+(2(10)-1)\\ &= 1+3+5+...+19\\ &= 100\end{aligned}$$

(β)

$$\begin{aligned}\sum_{n=1}^{4}\frac{1}{n+2} &= \frac{1}{1+2}+\frac{1}{2+2}+\frac{1}{3+2}+\frac{1}{4+2}\\ &= \frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}\\ &= \frac{20}{60}+\frac{15}{60}+\frac{12}{60}+\frac{10}{60}\\ &= \frac{20+15+12+10}{60}\\ &= \frac{57}{60}\end{aligned}$$

Problem Set Solutions

1. As a mathematician, you observe that there are some Greek letters that you just never encounter unless you work in very specific fields of math. The letters ι (iota), o (omicron), ν (nu), and υ

(upsilon) are some examples of uncommonly used Greek letters.

(a) Why might these Greek letters not be used as variables?

(b) What letters in the English alphabet would you not expect to be used as variables?

Solution: There is no one correct answer.

(a) Some Greek letters just look like English letters and thus it's hard to tell the difference. This might be the case for ν (nu) and υ (upsilon) which both resemble v. The letter o (omicron) could easily be mistaken as an o or 0. Also ι (iota) could easily be mistaken with a 1, since unlike i, there is no defining dot.

(b) For similar reasons as omicron, o is probably the worst letter for a variable since it looks like a 0. The letter s is also risky because it could be mistaken for a 5. Lastly z could be mistaken as a 2, but that hasn't stopped us from using it before (e.g. the z-axis).

2. Solve for the unknown variables.

(a) A circle has a diameter δ (delta). Given that the circumference of the circle is $C = 10\tau$ units, find the value of δ.

(b) A circle has a radius ρ. Given that the area of the circle is $A = 25\pi$ square units, find the value of ρ.

(c) **Challenge:** A right-angle triangle Δ has angles α, β and θ. Given that $\theta = 2\alpha$, what are some possible values of α, β, and θ? There is more than one possible answer.

Solution:

(a) Recall that the circumference of a circle with diameter d is given by $C = d\pi$. For this circle, $C = \delta\pi$. We can equate the formula to the given circumference and find

$$\begin{aligned}
\delta\pi &= 10\tau \\
\delta\pi &= 10(2\pi) \\
\delta\cancel{\pi} &= 10(2)\cancel{\pi} \\
\delta &= 20
\end{aligned}$$

Therefore $\delta = 20$ units.

(b) Recall that the area of a circle with radius r is given by $A = \pi r^2$. For this circle $A = \pi\rho^2$. We can equate the formula to the given area and find

$$\begin{aligned}
\pi\rho^2 &= 25\pi \\
\cancel{\pi}\rho^2 &= 25\cancel{\pi} \\
\rho^2 &= 25 \\
\sqrt{\rho^2} &= \sqrt{25} \\
\rho &= 5
\end{aligned}$$

Therefore $\rho = 5$ units.

(c) Recall that the sum of angles in a triangle is $180°$.

Solution 1: θ is the right-angle ($\theta = 90°$).

Solving $90° = 2\alpha$ gives $\alpha = 90° \div 2 = 45°$.

Then, $\beta = 180° - \theta - \alpha = 180° - 90° - 45° = 45°$.

Therefore the triangle Δ is isosceles with $\alpha = \beta = 45°$ and $\theta = 90°$.

Solution 2: β is the right-angle ($\beta = 90°$). Then

$$\begin{aligned}
180° &= \beta + \theta + \alpha \\
180° &= 90° + \theta + \alpha \\
180° - 90° &= \theta + \alpha \\
90° &= \theta + \alpha \\
90° &= 2\alpha + \alpha \\
90° &= (\alpha + \alpha) + \alpha \\
90° &= 3\alpha \\
\frac{90°}{3} &= \alpha \\
30° &= \alpha
\end{aligned}$$

Lastly $\theta = 2\alpha = 2(30°) = 60°$.

Therefore triangle Δ has angles $\alpha = 30°$, $\beta = 90°$, and $\theta = 60°$.

3. Recall that the prime-counting function, $\pi(x)$, outputs the number of prime numbers less than or equal to x.
   (a) Calculate the following.
      i. $\pi(30)$
      ii. $\pi(31)$
      iii. $\pi(32)$
   (b) Is it true that $\pi(17) = \pi(18)$?
   (c) Is it true for every prime number p that $\pi(p) = \pi(p + 1)$?

Solution:

(a) i. $\pi(30) = 10$
   ii. $\pi(31) = 11$
   iii. $\pi(32) = 11$

(b) We find that $\pi(17) = 7$ and $\pi(18) = 7$. Hence $\pi(17) = \pi(18)$.

(c) The equality $\pi(p) = \pi(p + 1)$ is not true for $p = 2$. This is because $\pi(2) = 1$ but $\pi(2 + 1) = \pi(3) = 2$.

However, $\pi(p) = \pi(p + 1)$ is true for any p larger than 2. All primes p larger than 2 are odd numbers. The number $p+1$ would be even, which is divisible by 2 and thus composite.

4. The Fundamental Theorem of Arithmetic states that we can write any positive integer as a unique product of its prime factors.

   For example $24 = 2 \times 2 \times 2 \times 3$, $50 = 2 \times 5 \times 5$. This product is also known as the **prime-factorization** of an integer.

   (a) Determine the prime-factorization of the following integers.
      i. 25
      ii. 36
      iii. 39
      iv. 40

   (b) We define $\Omega(n)$ to be the **Prime Omega Function** (since it's uppercase Omega). The function $\Omega(n)$ counts the number of prime factors of a positive integer n.

   For example, $\Omega(24) = 4$ and $\Omega(50) = 3$.

   For each integer in part (a), determine $\Omega(n)$.

(c) We define $\omega(n)$ to be the **prime omega function** (since it's lowercase omega). The function $\omega(n)$ counts the *unique* prime factors of a positive integer n. For example, $\omega(24) = 2$ since the unique prime factors are 2 and 3, we only count 2 once. The value of $\omega(50) = 2$ since we only count one of the 5's.

For each integer in part (a), determine $\omega(n)$.

(d) Consider the function given by $f(n) = \Omega(n) - \omega(n)$.

i. What can we say about n if $f(n) = 0$?

ii. What can we say about n if $f(n) \neq 0$?

Solution:

(a)
i. $25 = 5 \times 5$
ii. $36 = 2 \times 2 \times 3 \times 3$
iii. $39 = 3 \times 13$
iv. $40 = 2 \times 2 \times 2 \times 5$

(b)
i. $\Omega(25) = 2$
ii. $\Omega(36) = 4$
iii. $\Omega(39) = 2$
iv. $\Omega(40) = 4$

(c)
i. $\omega(25) = 1$
ii. $\omega(36) = 2$
iii. $\omega(39) = 2$
iv. $\omega(40) = 2$

(d) Answers may vary.

i. If $f(n) = 0$ then all of n's prime factors/divisors are unique.

ii. If $f(n) \neq 0$ then some of n's prime factors/divisors are repeated.

5. Rewrite each sum as an expression that uses Sigma Summation Notation. As a hint, each of the sums start at $n = 1$.

(a) $3 + 6 + 9 + 12 + 15 + 18 + 21 + 24 + 27 + 30$

(b) $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5}$

(c) $\frac{3}{2} + 2 + \frac{9}{4} + \frac{12}{5}$

(d) The sum of all positive even numbers.

(e) The sum of all positive odd numbers.

(f) $0 + 1 + 1 + 2 + 1 + 2 + 1 + 3 + 2 + 2$, (Hint: use a function from Question 4)

Solution:

(a) $3+6+9+12+15+18+21+24+27+30 = \sum_{n=1}^{10} 3n$

(b) $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5} = \sum_{n=1}^{4} \frac{n}{n+1}$

(c) $\frac{3}{2}+2+\frac{9}{4}+\frac{12}{5} = 3\left(\frac{1}{2}\right)+3\left(\frac{2}{3}\right)+3\left(\frac{3}{4}\right)+3\left(\frac{4}{5}\right) = \sum_{n=1}^{4} \frac{3n}{n+1}$

(d) $2+4+6+8+\cdots = \sum_{n=1}^{\infty} 2n$

(e) $1+3+5+7+\cdots = \sum_{n=1}^{\infty} (2n-1)$

Note, we could alternatively use $\sum_{n=0}^{\infty} (2n+1)$.

(f) $0+1+1+2+1+2+1+3+2+2 = \sum_{n=1}^{10} \Omega(n)$

Note, there might be another possible answer so share it on Piazza if you have one!

6. In this question, we will look at some shortcuts for certain sums. You may use a calculator if it helps.

(a) Evaluate the following sums.

i. $\sum_{n=1}^{6} 1$ ii. $\sum_{n=1}^{15} 1$ iii. $\sum_{n=1}^{75} 1$

(b) Predict the value of $\sum_{n=1}^{k} 1$, where k is any positive integer.

(c) Evaluate the following sums.

i. $\sum_{n=1}^{5} 4$ ii. $\sum_{n=1}^{9} 11$ iii. $\sum_{n=1}^{10} \pi$

(d) Predict the value of $\sum_{n=1}^{k} c$, where c is a constant and k is any positive integer.

(e) For any positive integer k, $\sum_{n=1}^{k} n = \frac{k(k+1)}{2}$. Use this formula to evaluate the following sums.

i. $\sum_{n=1}^{7} n$

ii. $\sum_{n=1}^{34} n$

iii. $\sum_{n=1}^{72} n$

Solution:

(a) i. $\sum_{n=1}^{6} 1 = 1+1+1+1+1+1 = 6$

ii. $\sum_{n=1}^{15} 1 = 1+1+\cdots+1 = 15$

iii. $\sum_{n=1}^{75} 1 = 75$

(b) $\sum_{n=1}^{k} 1 = k$

(c) i. $\sum_{n=1}^{5} 4 = 4+4+4+4+4 = 5(4) = 20$

ii. $\sum_{n=1}^{9} 11 = 11+11+\cdots+11 = 9(11) = 99$

iii. $\sum_{n=1}^{10} \pi = \pi+\pi+\cdots+\pi = 10\pi$

(d) $\sum_{n=1}^{k} c = ck$

(e) i. $\sum_{n=1}^{7} n = \frac{7(7+1)}{2} = \frac{7(8)}{2} = \frac{56}{2} = 28$

ii. $\sum_{n=1}^{34} n = \frac{34(34+1)}{2} = \frac{34(35)}{2} = \frac{1190}{2} = 595$

iii. $\sum_{n=1}^{72} n = \frac{72(72+1)}{2} = \frac{72(73)}{2} = \frac{5256}{2} = 2628$

7. On January 1st, 2022, Constantine did one pushup. The next day he did two pushups, and the day after he did three pushups.

   (a) Continuing this pattern of doing one more pushup than the day before, how many pushups in total did Constantine do in the month of January?

   (b) Assume that for each day of 2022, Constantine does his number of pushups according to the day of the month (i.e. on the first day of every month, he starts again at one pushup and increases the pushups by one each day until the end of the month). How many pushups in total will he do for 2022?

   (c) Constantine's sister, Constantina, does twice as many pushups as Constantine on any given day. How many pushups did she do in January?

Solution: Solutions may vary.

(a) We can express the number of pushups that Constantine did as a sum:

$$1+2+3+\cdots+31=\sum_{n=1}^{31} n$$

Using the formula given in Question 6(e), we have

$$\sum_{n=1}^{31} n=\frac{31(31+1)}{2}=\frac{31(32)}{2}=\frac{992}{2}=496$$

Therefore, Constantine did 496 pushups in the month of January.

(b) Not every month has the same number of days. We will first determine how many pushups he does in a month with 28 days (for February) and how many pushups he does in months with 30 days.

For February the number of pushups he does is as follows:

$$\sum_{n=1}^{28} n=\frac{28(28+1)}{2}=\frac{28(29)}{2}=\frac{812}{2}=406$$

Instead of doing the formula for months with 30 days, we can also subtract 31 from what we got in part (a) to get

$$\sum_{n=1}^{30} n=496-31=465$$

Now we just add up all the months. Note there are 7 months with 31 days, 4 months with 30 days and 1 month with 28 days. We have the total number of pushups as

$$\begin{aligned}
&496+406+496+465+496+465+496+496+465+496+465+496\\
&=(7)496+4(465)+406\\
&=3472+1860+406\\
&=5738
\end{aligned}$$

So by the end of 2022, Constantine will have done 5738 pushups.

(c) There are two ways we can do this question.

Solution 1: Most of you probably want to just take Constantine's January sum of 496 and multiply it by 2. This is correct and we find that Constantina did $496 \times 2 = 992$ pushups in January.

Solution 2: We could calculate a separate sum as follows:

$$2 + 4 + 6 + \cdots + 62 = \sum_{n=1}^{31} 2n$$

However we don't really have a formula for $\sum_{n=1}^{k} 2n$. If we assume that

$$\sum_{n=1}^{31} 2n = 2 \times \frac{31(31+1)}{2} = 31(32) = 992$$

you notice we still get the same answer.

Think about doubling every number and then summing them together. Then think about taking the sum first, and doubling the total after. They are equivalent in value.

This is a property of summations that we'll discuss more in the Live Session. More generally, for any constant c and any integer function $f(n)$, we have

$$\sum c \times f(n) = c \times \sum f(n)$$