UNIVERSITY OF WATERLOO

Faculty of Mathematics
Waterloo, Ontario N2L 3G1

Centre for Education in
Mathematics and Computing

Grade 6 Math Circles

Wednesday, March 3, 2021

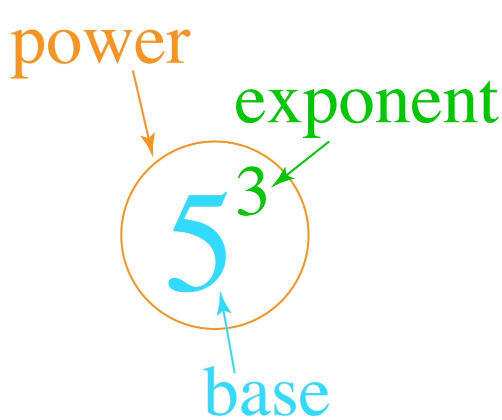
Exponents

Powers

Multiplication is often introduced as repeated addition e.g. $5 \times 3 = 5 + 5 + 5$. Similarly, **exponentiation** is repeated multiplication e.g. $5^3 = 5 \times 5 \times 5$. This operation makes it easier to write and use recurrent multiplications.

Exercise: Calculate 2×5 and 2^5. Are the answers the same? Is it possible for $2 \times n$ to equal 2^n where n is an integer? If so, what is a possible n?

Take a look at the picture below and observe the notation used for exponents.



The large number is called the **base** and the small number at the top is the **exponent**. The exponent indicates how many times to use the base in a multiplication. In the picture above with the example 5^3, the 3 signifies that 5 is being multiplied 3 times. It can be written as $5 \times 5 \times 5$. This method of writing exponents is called *expanded form*.

When we refer to the base and exponent together, we use the term **power**. In words, we can say that 5^3 is "5 to the exponent 3", "5 to the third power" or simply, "5 cubed". Similarly, when we have a base to the exponent 2 such as 7^2, we can refer to the power as 7 squared.

Example 1: Write the following powers in expanded form and then evaluate.

a) 6^4 b) 3^5 c) 8^2 d) 9^3

Solution:

a) $6^4 = 6 \times 6 \times 6 \times 6 = 1296$

b) $3^5 = 3 \times 3 \times 3 \times 3 \times 3 = 243$

c) $8^2 = 8 \times 8 = 64$

d) $9^3 = 9 \times 9 \times 9 = 729$

Special Bases & Exponents

The base and exponent of a power can be the number 0 or 1.

For any base to the exponent 1, we get the value of the base. So, 99^1 simplifies to 99. In other words, we are multiplying 99 by itself just once which is 99. This can be thought of in the same way that 99×1 means to add 99 to itself just once, which is again 99.

It is mathematical convention that any base to the exponent 0 is equal to 1. Using 99 as our base again, we see that $99^0 = 1$. We will show one explanation of this rule later on.

On the other hand, if 1 is our base, for any exponent we will get the value 1. Observe that 1 multiplied by itself equals 1, no matter how many times it is multiplied by itself. Similarly, when we have base 0, for any exponent except 0 itself, we will get the value 0.

Example 2: Evaluate the following powers.

a) 101^0 b) 0^{333} c) 8^1 d) 1^{94}

Solution:

a) $101^0 = 1$ b) $0^{333} = 0$ c) $8^1 = 8$ d) $1^{94} = 1$

Negatives in Powers

Since the base and exponent of a power can be any real number, and the set of real numbers includes integers (i.e. negative numbers), this means that we can have negative exponents and bases.

In the case of a base to a negative exponent, we simply take the **reciprocal** of the base to the corresponding positive exponent. The **reciprocal** of a number is easily found by calculating 1 divided by that number. For example, to evaluate 7^{-2}, we first take the reciprocal of 7 which is $1 \div 7 = \frac{1}{7}$. Then, we evaluate $\left(\frac{1}{7}\right)^2$. To take the power of a base that is a fraction, we take the power of the numerator and denominator. So, $\left(\frac{1}{7}\right)^2 = \frac{1^2}{7^2}$ or $\frac{1}{49}$.

When evaluating a power with a negative base, we first note if the exponent is even or odd. Let $(-b)$ be a negative base, x be an even exponent, and y be an odd exponent. A negative base to an even exponent will result in a positive value so, we can use the formula $(-b)^x = b^x$. On the other hand, a negative base to an odd exponent will lead to a negative result giving us the formula $(-b)^y = -(b^y)$. Let's look at an example.

Consider the power $(-5)^4$. Since the exponent is even, we can directly evaluate $(-5)^4$ as 5^4 which computes to 625. Now, look at $(-6)^3$. The exponent is odd so the result will be negative. Thus, $(-6)^3 = -(6^3)$. As 6^3 equals 216, we get that $(-6)^3 = -216$.

Expand: Watch this video to explore the rule of powers with negative bases: `https://youtu.be/Z2S7N2kuBBY`.

Example 3: Evaluate the following powers.

a) 8^{-4} b) $\left(\frac{1}{15}\right)^{-2}$ c) $(-10)^6$ d) $(-3)^{-5}$

Solution:

a) $8^{-4} = \left(\frac{1}{8}\right)^4 = \frac{1^4}{8^4} = \frac{1}{4096}$ b) $\left(\frac{1}{15}\right)^{-2} = 15^2 = 225$

c) $(-10)^6 = 10^6 = 1,000,000$

d) $(-3)^{-5} = -(3^{-5}) = -\left(\frac{1}{3}\right)^5 = -\left(\frac{1^5}{3^5}\right) = -\frac{1}{243}$

Exponent Rules

To perform operations involving exponents, there are a few rules. Let b be a base and let m and n be exponents. Then, we get the following rules.

Product Rule: When multiplying two powers with the same base, add the exponents.

$$b^m \times b^n = b^{m+n}$$

For example, $8^7 \times 8^9 = 8^{7+9} = 8^{16}$.

Quotient Rule: When dividing two powers with the same base, subtract the exponents.

$$\frac{b^m}{b^n} = b^{m-n}$$

For example, $\frac{13^{10}}{13^5} = 13^{10-5} = 13^5$.

Power Rule: When raising a power to an exponent, multiply the exponents.

$$(b^m)^n = b^{m\times n}$$

For example, $(20^3)^6 = 20^{3\times 6} = 20^{18}$.

Try This: Practice using the exponent rules with this GeoGebra app: https://www.geogebra.org/m/dm76f52n.

Explore: Bases to the Exponent 0

Now, we can return to the rule that for any base, b, where b is not equal to 0, $b^0 = 1$. Consider the expression $\frac{b^x}{b^x}$ where x is an exponent. Using the quotient rule, we get that $\frac{b^x}{b^x} = b^{x-x} = b^0$. At the same time, note that a fraction where the numerator and denominator are the same simplifies to 1. So,

$$\frac{b^x}{b^x} = 1 \Rightarrow b^0 = 1$$

How about when our base is 0? Use a calculator to evaluate 0^0. What do you get? The calculator should show you a math error. Most mathematicians consider 0^0 to be an **indeterminate form**, or say that the statement is **undefined**. Others define $0^0 = 1$. It has been long debated in mathematics and is considered **ambiguous**.

We can use a similar explanation as above to show why there is uncertainty about the value of 0^0. If we rewrite 0^0 as a fraction, $\frac{0^x}{0^x}$, both numerator and denominator will simplify to 0 for any exponent x not equal to 0. Then, we have the fraction $\frac{0}{0}$ which is an **indeterminate form**.

Exponential Growth and Decay

From medicine to computers, many fields use exponents. In this section, we are going to look at a few applications of exponents and powers.

Example 4: A population of rabbits doubles every month. If there were 15 rabbits at the start of the year, how many rabbits will there be after 6 months?

Solution: Note that the population is doubling every month which means that we are multiplying the population by 2 each month. We can create the following table to organize the population growth.

Month	Rabbit Population
0	15
1	30
2	60
3	120
4	240
5	480
6	960

Therefore, there will be 960 rabbits after 6 months. That's a lot of rabbits!

Alternatively, we could calculate the rabbit population by first noting that the population will be doubling 6 times, once for every month. Since doubling is the same as multiplying by 2, we can say that the population after 6 months is equal to our initial population multiplied by 2, six times. This can be written as $15 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$. We can rewrite the equation using our knowledge of exponents. So,

$$\begin{aligned}
\text{rabbit population} &= 15 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \\
&= 15 \times 2^6 \\
&= 15 \times 64 \\
&= 960
\end{aligned}$$

However, not all things grow at an exponential rate. Some decay or decrease exponentially.

Example 5: A special edition baseball card is initially worth \$200. It is predicted to lose 2.5% of its value every year for the next 10 years. Sham bought 5 baseball cards when they were first launched. If he tries to sell them after 4 years, how much can he sell them for?

Solution: First, we calculate how much all 5 cards were worth at the beginning. Since each card is worth \$200, together they are valued at \$1000. Note that if they lose 2.5% of their value every year, the cards are now worth 97.5% of their original value at the end of each year. In other words, we are multiplying the value of the cards by 0.975 each year to figure out their new value. Let's create a table to organize our calculations, rounding to 2 decimal places.

Year	Value of Cards (\$)
0	1000
1	975
2	950.63
3	926.86
4	903.69

Therefore, he can sell the cards for \$903.69 after 4 years. Observe that the baseball cards decreased in value the most after the first year. Every consecutive year they decreased less and less.

Here is a more concise solution: Recall that the cards retain 97.5% of their value. Since we are multiplying the initial value of the cards by 0.975, four times, this can be written as $1000 \times 0.975 \times 0.975 \times 0.975 \times 0.975$. Using exponents to simplify the equation, we get

$$\begin{aligned} \text{value} &= 1000 \times 0.975 \times 0.975 \times 0.975 \times 0.975 \\ &= 1000 \times 0.975^4 \\ &= 903.6878906... \\ &\approx 903.69 \end{aligned}$$

This solution is more accurate than the solution above as we are not rounding every time and instead, approximating only once at the very end.