Grade 6 Math Circles  
Winter 2018 - February 13/14  

Graph Theory

Warm-up!
Imagine you are the manager of your favourite sports team or singer. You need to schedule their route so that they can travel as little as possible, starting and ending at Colorado Springs. What is the shortest path?

Today we will study graph theory. We will learn some definitions in order to figure out what graph theory is!

Definitions
A vertex is a point. When we have more than one, we call them vertices and denote them by $v$.

An edge is a line segment that joins two vertices, denoted by $e$.

When two vertices are connected by an edge, we say that the vertices are adjacent.

The endpoints of an edge are the vertices at either end.

A vertex is incident to an edge if it is that edge’s endpoint.
A graph $G$ is a set of vertices and edges. We write $G=(V,E)$.
Graphs can represent networks (think about a connection of someone’s Facebook friends!),
a map or road system (see the warm-up), or any connection between different objects!

The neighbourhood of a vertex is all of the vertices adjacent to that vertex. Two vertices in a neighbourhood are called neighbours.

From the following graph, highlight the edges and circle the vertices. List vertices that are adjacent with each other. Find the neighbourhood of the vertex $f$.

A walk is a sequence of vertices where each vertex in the walk is adjacent.
A path is a walk without repeated vertices.
A cycle is a path that starts and ends at the same vertex.
From the following graph, find a walk from A to B and a path from A to B, as well as a cycle from B.

Bipartite Graphs
We say that a graph is bipartite if the vertices of the graph can be split into two sets, where no two vertices from either set are adjacent.

Are the following graphs bipartite? If so, label the vertices of one set “A” and the vertices of the other “B”. If not, explain why.

Draw a planar, non-bipartite graph.
Colouring

When we colour a graph or give it an n-colouring (n denoting the minimum number of colours needed for the graph), we label the vertices so that no two adjacent vertices share the same colour.

Recall that we can colour bipartite graphs using just 2 colours! So bipartite graphs are 2-colourable. Colour the following graphs:

Notice that maps are just graphs without vertices: they are connected at the edges! Now colour the following maps. Use the least amount of colours possible.

What is the least amount of colours we can use to colour a map?
The ____-colour theorem states that we can colour any map using at most ____ colours!
Weighted Graphs and Minimum Spanning Trees

A **weighted graph** is a graph that assigns a numerical weight to each edge. If you have ever played the game *Ticket to Ride*, you can think of weights as the number of trains between each city.

---

A **tree** is a graph with no cycles. A **spanning tree** is a tree that contains all vertices of a graph. A **minimum spanning tree** is a spanning tree that has the smallest total weight (found by adding up all of the edge weights in the spanning tree) compared to any other spanning trees.

For example, consider the following graph:

1. Find and highlight two examples of a tree in the graph.
2. Find and highlight two spanning trees in the graph.
To find a minimum spanning tree for a graph, we use the following algorithm:

**Prim’s Algorithm**

1. Select a starting vertex. It is now in the tree.
2. For any vertices in the tree, select the neighbour incident to the edge with the smallest weight and add it to the tree.
3. Continue until all vertices are in the tree. This is the minimum spanning tree!

Using Prim’s Algorithm, find a minimum spanning tree from the following graphs:
Now, consider the Traveling Salesman Problem (from the warm-up).
There is no algorithm to solve the problem - if you find one you could be a millionaire!
If we consider a variation on the problem as a minimum spanning tree problem, we could use Prim’s algorithm to solve it (we can’t return to the starting spot with a tree though!):
Problems
1. For the following graphs,
   a) list the edges and vertices of the graphs
   b) list the neighbours of w
   c) How many edges are incident with s?
   d) Find a walk between s and t. Is your walk a path?
   e) Is the graph a tree? If not, find and highlight a cycle.

2. Are the following graphs planar? Why or why not?

3. If a graph has 4 edges and 3 vertices, could it be a planar graph?
4. If a graph has 3 edges and 4 vertices, could it be a planar graph?
5. Can you walk across each bridge exactly once in the following picture?

6. Each of House 1, House 2 and House 3 needs to get water, gas and electricity. The challenge is for you to connect each house to each utility. The catch: No lines can cross! Is it possible?
7. Are the following graphs bipartite? If so, label the vertices to indicate the bipartition.

8. Evan is building the new high speed train route around the Waterloo area. He has a map with the stops and the distance between each stop. Where should Evan build the tracks to minimize the cost of construction yet ensure one can get between each stop on the train?
9. Colour the following graphs or maps with as few colours as possible. State what this number is below the map.

10. A matching of a graph is a set of edges where no edges in the set share an endpoint. The size of a matching is the number of edges in the set. Find the maximum matching for the following graphs:
11. For the following graph:
   a) Give an example of a walk from A to B using repeated vertices.
   b) Give an example of a path from A to B.
   c) Give an example of a cycle.
12. The degree of a vertex is the number of edges incident to it.

The **Handshaking Lemma** states that for any graph, the sum of all degrees in the graph (for all vertices) is equal to two times the number of edges in the graph.

For example, consider the following graph:

![Graph Diagram]

Each vertex has 3 degrees and there are 6 vertices, so the sum of degrees is $3 \times 6 = 18$.

There are 9 edges in total.

Therefore, the sum of degrees is equal to twice the number of edges since $9 \times 2 = 18$.

Prove the Handshaking Lemma.