Last week we discussed linear Diophantine equations (LDEs), which are equations of the form

\[ ax + by = c \]

for some integers \(a, b,\) and \(c\). The important thing about Diophantine equations is that their solutions have to be integers!

Here are some examples from last time:

(I) \(8x + 3y = 10\) has a solution given by \(8(-10) + 3(30) = 10\).

(II) \(14x + 35y = 4\) has no solutions. Why? Because if we divide both sides by 7, we get

\[ 2x + 5y = \frac{4}{7}. \]

Since the left-hand side is an integer and the right-hand side is not, no solutions can exist.

Example (II) shows that the existence of solutions to \(ax + by = c\) depends on the common divisors of \(a\) and \(b\). In particular, it was important for us to be able to find \(\gcd(a, b)\), the greatest common divisor of \(a\) and \(b\). To do this, we used the following tools:

**The Division Algorithm.** Let \(a\) and \(b\) be integers with \(b > 0\). There are unique integers \(q\) (the quotient) and \(r\) (the remainder) such that

\[ a = bq + r \quad \text{and} \quad 0 \leq r < b. \]

Furthermore, \(\gcd(a, b) = \gcd(b, r)\).
The Euclidean Algorithm.

Step 1: Arrange $a$ and $b$ so that $a \geq b$.

Step 2: Write $a = bq + r$ where $0 \leq r < b$.

Step 3: If $r = 0$, then stop! We get $\gcd(a, b) = \gcd(b, 0) = b$.

Step 4: Replace $(a, b)$ with $(b, r)$ and return to Step 1.

These algorithms led to the following result on solutions to $ax + by = c$:

**Theorem.** The LDE $ax + by = c$

has a solution if and only if $\gcd(a, b)$ divides $c$.

Using the Euclidean algorithm and working backwards, we get an equation of the form $ax_0 + by_0 = \gcd(a, b)$. The solution to the LDE can then be obtained by multiplying both sides of this equation by $\frac{c}{\gcd(a, b)}$. So we have $ax + by = c$ where

$$x = x_0 \cdot \frac{c}{\gcd(a, b)} \quad \text{and} \quad y = y_0 \cdot \frac{c}{\gcd(a, b)}.$$

**Example.** Use the above approach to solve the LDE $66x + 15y = 18$.

**Solution.** We’ll start by using the Euclidean algorithm to find $\gcd(66, 15)$:

$$66 = 15(4) + 6 \quad \Rightarrow \quad \gcd(66, 15) = \gcd(15, 6) \quad (1)$$

$$15 = 6(2) + 3 \quad \Rightarrow \quad \gcd(15, 6) = \gcd(6, 3) \quad (2)$$

$$6 = 3(2) + 0 \quad \Rightarrow \quad \gcd(6, 3) = \gcd(3, 0) = 3.$$
Since \( \gcd(66, 15) = 3 \) divides 18, there is a solution to this LDE. To find it, we work backwards through the Euclidean algorithm:

\[
3 = 15 - 6(2) \quad \text{by (2)}
\]
\[
= 15 - [66 - 15(4)](2) \quad \text{by (1)}
\]
\[
= 15(9) + 66(-2)
\]

Thus, we arrive at the equation \( 66(-2) + 15(9) = 3 \). A solution to our LDE can be obtained by multiplying the above expression by \( 18/3 = 6 \):

\[
66(-12) + 15(54) = 18.
\]

1 Finding the Complete Solution to an LDE

Let’s consider the following variation on the Mario example from last time:

**Example (Goomba’s revenge!).** Goomba has forgotten how to run, and instead can only jump forward or backward. He can make long jumps 6 metres in length, or short jumps 4 metres in length. Mario (who has also forgotten how to run) stands 16 metres away.

If Goomba makes \( x \) long jumps and \( y \) short jumps, find all possible choices of \((x, y)\) that will allow him to land on Mario.

Just like last time, this problem can be represented as a linear Diophantine equation:

\[
6x + 4y = 16.
\]

By using the Euclidean algorithm and working backwards, we can find a particular solution:

\[
6(8) + 4(-8) = 16.
\]

So he will land on Mario by making 8 long jumps forward and 8 short jumps backward. But this is not the only way!

**Provide at least 2 more solutions.**
Uh oh... there may be quite a few solutions. How many more are there? How do we obtain them? To answer these questions, we’re going to have to think cleverly!

By moving things around a bit, the equation can be written as

\[
6x + 4y = 16 \quad \rightarrow \quad 4y = -6x + 16 \quad \rightarrow \quad y = \frac{-6}{4}x + \frac{16}{4} \quad \rightarrow \quad y = -\frac{3}{2}x + 4
\]

We recognize this as the equation of a line with slope \( = \) \( -\frac{3}{2} \) and y-intercept \( = \) \( -4 \).

The solutions to our LDE are exactly the points \((x, y)\) on this line whose coordinates are integers!

On the left is the graph of the line \(6x + 4y = 16\).

Below are some of the integer points on this line:

<table>
<thead>
<tr>
<th>(x)</th>
<th>(y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\cdot)</td>
<td>(\cdot)</td>
</tr>
<tr>
<td>(-2)</td>
<td>7</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>(-2)</td>
</tr>
<tr>
<td>6</td>
<td>(-5)</td>
</tr>
<tr>
<td>8</td>
<td>(-8) (\leftarrow) our solution!</td>
</tr>
<tr>
<td>10</td>
<td>(-11)</td>
</tr>
<tr>
<td>(\cdot)</td>
<td>(\cdot)</td>
</tr>
</tbody>
</table>

These are the solutions to our LDE!

Do you notice a pattern?
Suppose we move along this line from top-left to bottom-right, starting at a particular solution \((x_0, y_0)\).

To get to the next solution, we must add ____ to the \(x\)-value and subtract ____ from the \(y\)-value.

To get to the previous solution, we must subtract ____ from the \(x\)-value and add ____ to the \(y\)-value.

The following theorem summarizes this phenomenon:

**Theorem.** Suppose that \((x_0, y_0)\) is one solution to the linear Diophantine equation \(ax + by = c\), and let \(d = \gcd(a, b)\). Then the full list of integer solutions is given by

\[
x = x_0 + n \left( \frac{b}{d} \right), \quad y = y_0 - n \left( \frac{a}{d} \right)
\]

where \(n\) is any integer.

Let’s test out this theorem for \(6x + 4y = 16\), the LDE from our previous example. We found that

- \(a = 6\), \(b = 4\),
- \(d = \gcd(6, 4) = 2\), and
- \((x_0, y_0) = (8, -8)\) is a particular solution.

This means that the full list of solutions is given by

\[
x = x_0 + n \left( \frac{b}{d} \right) = 8 + n \left( \frac{4}{2} \right) = 8 + 2n
\]

\[
y = y_0 - n \left( \frac{a}{d} \right) = -8 - n \left( \frac{6}{2} \right) = -8 - 3n
\]

where \(n\) is any integer.
In the table below, we plug in various values for $n$. Each time we do, we get a different solution $(x, y)$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>−1</th>
<th>−2</th>
<th>−3</th>
<th>−4</th>
<th>101</th>
<th>…</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x = 8 + 2n$</td>
<td>8</td>
<td>10</td>
<td>___</td>
<td>14</td>
<td>16</td>
<td>6</td>
<td>___</td>
<td>2</td>
<td>0</td>
<td>___</td>
<td>…</td>
</tr>
<tr>
<td>$y = −8 − 3n$</td>
<td>−8</td>
<td>−11</td>
<td>−14</td>
<td>−17</td>
<td>___</td>
<td>−5</td>
<td>−2</td>
<td>1</td>
<td>4</td>
<td>___</td>
<td>…</td>
</tr>
</tbody>
</table>

How many solutions does our LDE have? ____________________________.

**Example.** Find all integer solutions to the LDE

$$66x + 15y = 18.$$  

**Solution.** This was the example from before. We saw that

- $a = 66$, $b = 15$,
- $d = \gcd(a, b) = 3$, and
- $(x_0, y_0) = (−12, 54)$ is a particular solution.

Thus, the full list of solutions is given by

<table>
<thead>
<tr>
<th>$x$</th>
<th>___</th>
<th>___</th>
<th>___</th>
<th>___</th>
<th>___</th>
<th>___</th>
<th>___</th>
<th>___</th>
<th>___</th>
<th>___</th>
<th>…</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>___</td>
<td>___</td>
<td>___</td>
<td>___</td>
<td>___</td>
<td>___</td>
<td>___</td>
<td>___</td>
<td>___</td>
<td>___</td>
<td>…</td>
</tr>
</tbody>
</table>
2 Restrictions on Solutions to LDEs

Recall the following example from last time:

**Example.** Becky needs $2.15 to buy an extra large coffee. She only has quarters and dimes, and the cashier insists that she pay with exact change.

Is there a combination of quarters and dimes that will total $2.15?

**Solution.** This example is asking for a solution to the LDE

\[ 25x + 10y = 215. \]

By using the Euclidean algorithm and working backwards, we obtain the particular solution

\[ 25(43) + 10(-86) = 215. \]

It looks like Becky can buy her coffee with 43 quarters and \(-86\) dimes.

Wait... there’s something wrong here: our solution doesn’t make sense in the context of this problem. What we really want is a solution where both \(x\) and \(y\) are integers.

To accomplish this, we’ll need to find all solutions to the LDE, and then restrict our attention to the ones that make sense. Since

- \(a = 25, \ b = 10,\)
- \(d = \gcd(a, b) = 5,\) and
- \((x_0, y_0) = (43, -86)\) is a particular solution,

so the full list of solutions is given by

\[ x = x_0 + n \left( \frac{b}{d} \right) = 43 + n \left( \frac{10}{5} \right) = 43 + 2n \]
\[ y = y_0 - n \left( \frac{a}{d} \right) = -86 - n \left( \frac{25}{5} \right) = -86 - 5n \]

where \( n \) is any integer.

For \((x, y)\) to be a valid solution, we need both \(x \geq 0\) and \(y \geq 0\). Which values of \( n \) will satisfy these conditions?

\[
x \geq 0 \quad \Rightarrow \quad 2n \geq -43
\]
\[
\Rightarrow \quad n \geq -\frac{43}{2} = -21.5
\]
\[
\Rightarrow \quad n \geq -21 \quad \text{(since } n \text{ is an integer}).
\]

\[
y \geq 0 \quad \Rightarrow \quad 5n \leq -86
\]
\[
\Rightarrow \quad n \leq -\frac{86}{5} = -17.2
\]
\[
\Rightarrow \quad n \leq -18 \quad \text{(since } n \text{ is an integer}).
\]

Below are the 4 valid solutions to this equation:

<table>
<thead>
<tr>
<th>( n )</th>
<th>-18</th>
<th>-19</th>
<th>-20</th>
<th>-21</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x = 43 + 2n )</td>
<td>7</td>
<td>5</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>( y = -86 - 5n )</td>
<td>4</td>
<td>9</td>
<td>14</td>
<td>19</td>
</tr>
</tbody>
</table>

**Remark.** Solving for \( n \) in an inequality is just like solving for \( n \) in an equation. The only difference: we must reverse the inequality sign when multiplying or dividing by a negative number.