

SET THEORY - ASSIGNMENT 2

Functions.

- (1) For each function, determine if it is injective, surjective, bijective, or none of the above.
- (a) $f : \{1, 2, 3, 4\} \rightarrow \{5, 6, 7\}$, such that $f(1) = 7, f(2) = 6, f(3) = 7, f(4) = 6$.
 - (b) $g : \mathbb{Z} \rightarrow \mathbb{Z}, g(a) = 2a$.
 - (c) $h : \mathbb{N} \rightarrow \mathbb{N}$, such that $h(a)$ is the number of prime numbers smaller than a .
 - (d) $k : [0, 3] \rightarrow [3, 9], k(x) = 2x + 3$.
 - (e) $p : \mathbb{R}^3 \rightarrow \mathbb{R}^2, p(x, y, z) = (x, y)$.
 - (f) $i : A \rightarrow \mathcal{P}(A), i(a) = \{a\}$, where A is some set.

Solution. Neither, injective, surjective, bijective, surjective, injective.

- (2) Find a function $f : [0, 2] \rightarrow [0, 1]$ that is surjective but not injective, a function that is injective but not surjective, a function that is neither, and a function that is both.

Solution. Surjective:

$$f(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 1 & \text{otherwise} \end{cases}$$

Injective: $f(x) = \frac{x}{4}$.

Both: $f(x) = \frac{x}{2}$.

Neither:

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } 0 \leq x \leq 1 \\ 1 & \text{otherwise} \end{cases}$$

- (3) Suppose that we have surjective functions $f : A \rightarrow B$ and $g : B \rightarrow C$. Find a surjective function from A to C .

Solution. Define a function $h : A \rightarrow C$ by $h(x) = g(f(x))$. To see that h is surjective, let $z \in C$. Since g is surjective, there is $y \in B$ such that $g(y) = z$. Since f is surjective, there is $x \in A$ such that $f(x) = y$. Therefore, $h(x) = g(f(x)) = g(y) = z$.

Cardinalities.

- (1) For each of the following pairs of sets, determine whether they have the same cardinality. If not, find the bigger set.
- (a) $A =$ the set of functions from $\{1, 2, 3\}$ to $\{1, 2\}$, $B = \{1, 2, 3\} \times \{1, 2\}$.

Solution. $|A| = 8, |B| = 6$. Therefore $|A| > |B|$.

- (b) $A =$ the students in the class, $B =$ the chairs in the class.

Solution. Every student in the class got their own seat. This defines an injective function from the students to the chairs. Therefore, the number of chairs is bigger than the number of students.

(c) A = the rational numbers whose denominator is a multiple of 7, $B = \mathbb{Q}$.

Solution. A is contained in \mathbb{Q} , therefore, $|A| \leq \aleph_0$. On the other hand, the map $f : A \rightarrow \mathbb{N}$, $f(\frac{a}{b}) = a$ is a surjective map from A to \mathbb{N} , so $|A| \geq \aleph_0$, and it follows that $|A| = \aleph_0$. Since $|B| = \aleph_0$ as well, the two sets have the same cardinality.

(2) In Hilbert's hotel, each room is labeled with a natural number, $0, 1, 2, 3, \dots$. An infinite group of people comes in, in which every person is labeled with a real number between 0 and 1 with a finite presentation. That is, the sequence of digits after the decimal point doesn't go on forever. Assuming that the hotel was empty before, show that each person can be assigned a different room. What if the hotel was full when the group came in?

Solution. Every person in the group can be identified with a distinct number of the form $0.d$, where d is a natural number. Therefore, if A is the group of people, then the function $f : A \rightarrow \mathbb{N}$, $f(0.d) = d$ is an injective function from the group of people to the rooms, and in particular assigns a different room for each person. For the second part, first move every person in room a to room $2a$, thus clearing all the odd numbered room. Now the function $f(0.d) = d \cdot 2 - 1$ assigns a different room for each person.

(3) Let A and B be sets (either finite or infinite). Show that $|A \cap B| \leq |A| \leq |A \cup B|$.

Solution. We need to find injective functions from $A \cap B$ to A and from A to $A \cup B$. But this is easy, just take the identity functions $f(x) = x$.

(4) Show that for every set A , $|A| \leq |\mathcal{P}(A)|$.

Solution. Just take the function $f(a) = \{a\}$. If $a \neq b$ then $\{a\} \neq \{b\}$, so f is injective.

Higher cardinalities.

(1) Find an explicit surjection from $\mathcal{P}(\mathbb{N})$ to \mathbb{N} .

Solution. Since $\mathcal{P}(\mathbb{N})$ contains the singleton set $\{a\}$ for each natural number a , any function that takes $\{a\}$ to a would be surjective. We just need to figure out what to do with the other members of $\mathcal{P}(\mathbb{N})$. We can, for instance, send all of them to 1: define

$$f(x) = \begin{cases} a & \text{if } x = \{a\} \text{ for some } a \in \mathbb{N}. \\ 1 & \text{otherwise.} \end{cases}$$

(2) Find a surjection from $[0, 1]$ to \mathbb{R} . Conclude that $|[0, 1]| = \aleph$.

Solution. There are many ways of solving this. One of them is using the trigonometric function $\arctan(x)$. Here is a more elementary way. For each real number between 0 and 1, let s be the sequence of the digits in the odd places in the

decimal expansion, and let t be the sequence of digits in the even places. For instance, if $x = 0.123455555555\dots$, we have $s = 1355555\dots$ and $t = 245555\dots$

Let A be the subset of $[0, 1)$ where s has a finite presentation (that is, becomes only zeroes after some time). Define $f : A \rightarrow \mathbb{R}$, such that a number x goes to the real number $s.t$. Just as in the question with the hotel above, the s part of the number hits every natural number. Since t can be any sequence, the t part ends up hitting any decimal expansion. Therefore, f is surjective. Now extend f to all of $[0, 1]$ by having it be constantly 0 on any number not in A .

It follows that $[0, 1] \geq \aleph$. Since $[0, 1] \subseteq \mathbb{R}$, it follows that also $[0, 1] \geq \aleph$.

(3) In this question, we will show that $2^{\aleph_0} = \aleph$.

(a) Let $A = \{0, 1\} \times \{0, 1\} \times \{0, 1\} \times \{0, 1\}$ (in other words, A consists of sequences of length four of zeros and ones). Find a surjection from A to the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Solution. A contains 16 elements, so there is a surjection $f : A \rightarrow \{0, 1, \dots, 9\}$.

(b) Show that every subset of \mathbb{N} can be represented by an infinite sequence of zeros and ones. Conclude that $\mathcal{P}(\mathbb{N})$ is isomorphic to the collection such sequences.

Solution. To every sequence of zeros and ones, assign the subset of \mathbb{N} consisting of all the positions of the 1's in the sequence (where we start counting from zero). For instance, the sequence 10110100000... goes to $\{0, 2, 3, 5\}$. Call this function g .

(c) Use the first two parts to find a surjection from $\mathcal{P}(\mathbb{N})$ to the collection of infinite sequences of $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Conclude that $2^{\aleph_0} \geq |[0, 1]|$.

Solution. g provides a surjection from $\mathcal{P}(\mathbb{N})$ to the infinite sequences of zeros and ones. If we apply f to every four digits in the sequence, we obtain a surjection to the infinite sequences of numbers $\{0, 1, \dots, 9\}$. By mapping each such sequence d to the number $0.d$, we obtain a surjection onto $[0, 1]$.

(d) Find a surjection from $[0, 1]$ to $\mathcal{P}(\mathbb{N})$, and conclude that $2^{\aleph_0} = |[0, 1]|$.

Solution. Let B be the subset of $[0, 1]$ of numbers of the form $0.d$, where d is a sequence of zeroes and ones. To each element in B , assign the set of indices of all the 1's in the sequence. For instance, the number 0.111010101 goes to the set $\{0, 1, 2, 4, 6, 8\}$. This provides a surjection from B to $\mathcal{P}(\mathbb{N})$, so $|B| \geq |\mathcal{P}(\mathbb{N})|$. Since $B \subset [0, 1]$, it follows that $|[0, 1]| \geq |\mathcal{P}(\mathbb{N})| = 2^{\aleph_0}$, and therefore $|[0, 1]| = |\mathcal{P}(\mathbb{N})| = 2^{\aleph_0}$.

(e) Use question ?? to conclude that $2^{\aleph_0} = \aleph$.

Solution. From Question ??, we know that $|[0, 1]| = \aleph$. Consequently, we conclude that $|2^{\aleph_0}| = |\mathcal{P}(\mathbb{N})| = \aleph$.

(4) If α and β are cardinal numbers (finite or infinite), then $\alpha + \beta$ is defined as follows. Choose sets A and B of cardinality α and β respectively, such that $A \cap B = \emptyset$. Then $\alpha + \beta$ is defined to be the cardinality of $A \cup B$.

- (a) Show that with this definition, $3 + 4 = 7$.

Solution. Choose $A = \{1, 2, 3\}$, $B = \{4, 5, 6, 7\}$. Then $|A \cup B| = |\{1, 2, 3, 4, 5, 6, 7\}| = 7$.

- (b) Show that $\aleph_0 + \aleph_0 = \aleph_0$ and $\aleph_0 + \aleph = \aleph$.

Solution. For the first part, choose A to be the odd integers, and B to be the even integers. Then $A \cap B = \emptyset$, $|A| = |B| = \aleph_0$, and $|A \cup B| = |\mathbb{Z}| = \aleph_0$. For the second part, choose $A = \mathbb{N}$, and $B = [-5, -4]$.

Remark. Can you show that the definition above does not depend on the choice of sets? That is, If instead of B we chose a different set B' such that $B' \cap A = \emptyset$ and $|B'| = |B|$, so we have $|A \cup B'| = |A \cup B|$?

- (5) Let A be the collection of functions from \mathbb{R} to \mathbb{N} . Show that $|A| > \aleph$.

Solution. Let B be the subset of A consisting of the functions that only hit 0 and 1. Each element in B can be identified with a subset of \mathbb{R} as follows. Let $f \in B$. Assign to f the set

$$S_f = \{x \in \mathbb{R} | f(x) = 1\}.$$

Since any subset of \mathbb{R} arises this way, the assignment $f \rightarrow S_f$ is a surjection, and therefore $|B| \geq \mathcal{P}(\mathbb{R})$. From Cantor's theorem $\mathcal{P}(\mathbb{R}) > \aleph$. Since $B \subseteq A$, it follows that $|A| > \aleph$ as well.

A confusing question.

- (1) The purpose of this question is to show that the set of all sets doesn't really exist. Assume that there was a set of all sets, call it S . Show that $|S| > |S|$ (hint: use a similar argument to the one used in the proof that $|\mathcal{P}(A)| > |A|$).

Solution. We can actually directly rely on the fact that $|\mathcal{P}(A)| > |A|$ for any set A , rather than using a similar argument as the one used in that proof.

Consider the set $\mathcal{P}(S)$. On one hand, all the elements of $\mathcal{P}(S)$ are sets, so $\mathcal{P}(S) \subseteq S$, and $|\mathcal{P}(S)| \leq |S|$. On the other hand, from Cantor's theorem we know that for EVERY set, $\mathcal{P}(S) > |S|$. So we conclude that $|S| > |S|$, which cannot be.