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Mathematics and Computing

Grade 6 Math Circles

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Cool Geometry

Introduction

This week, we'll be looking at a few interesting geometry facts. In the first half, we'll be covering two important theorems in geometry and their proofs.

The second half will consider some cool shapes in geometry that aren't typically covered in school. We'll also talk briefly about some interesting applications.

The Pythagorean Theorem

The Pythagorean Theorem is one of the most useful and important things you'll learn in math. It's been known for thousands of years and is typically attributed to **Pythagoras**, a Greek philosopher (Figure 1).

Figure 1: A bust of Pythagoras

The Pythagorean Theorem relates the sides of a right-angled triangle. It says that in a right-angled triangle, the sum of the squares of the base and the height is equal to the square of the longest side (or **hypotenuse**) (see Figure 2).

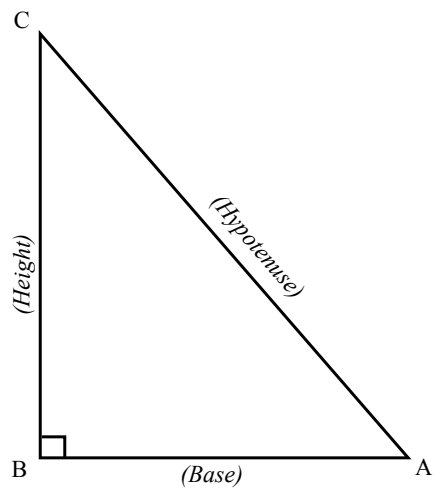

Figure 2: A right-angled triangle

In other words, the Pythagorean theorem states that in the triangle ABC shown above:

$$(Base)^2 + (Height)^2 = Hypotenuse^2$$

We can rewrite this in terms of the hypotenuse:

$$\sqrt{(Base)^2 + (Height)^2} = Hypotenuse$$

This theorem is used a lot in science - finding the distance between two points, designing lenses etc. Since this theorem is so useful, it might make sense to prove it mathematically.

A Proof of the Pythagorean Theorem

A *proof* in mathematics refers to a precise logical argument that you use to argue for the truth of some statement or proposition. The Pythagorean Theorem was simply given to you earlier - how do you know this is true?

We're going to give a well known proof of the Pythagorean theorem. We will prove this theorem by using pictures. We call a the base, b the height and c the hypotenuse of the right triangle. Let's arrange 4 of the same right triangles to get the figure below (Figure 3):

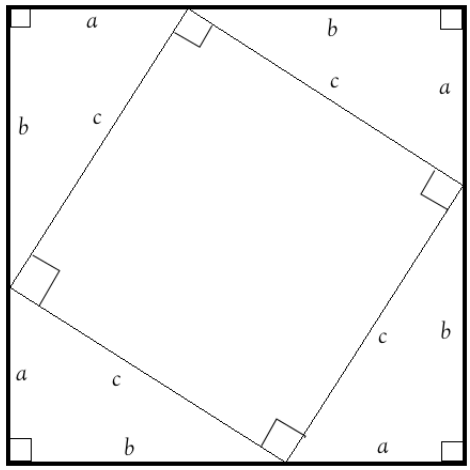

Figure 3: 4 identical right-angled triangles

Well, if the 4 side lengths of the figure in the middle are all c, this tells us that we have a square. The shaded region below represents the area of the square in the middle of our figure (Figure 4). This area is equal to c^2.

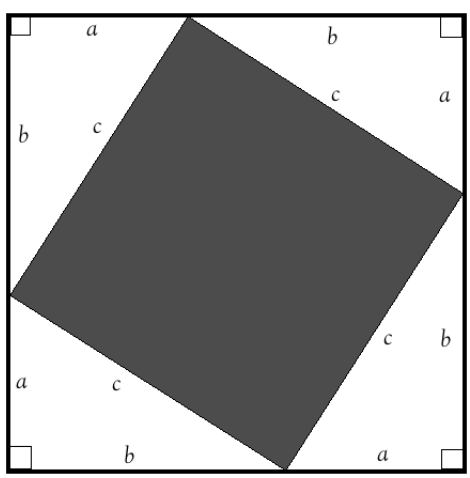

Figure 4: The area of the square in the center is equal to c^2

Now, remember that we need to show that $a^2 + b^2 = c^2$. So, because we can, let's play around with the arrangement of the 4 right triangles in the previous figure:

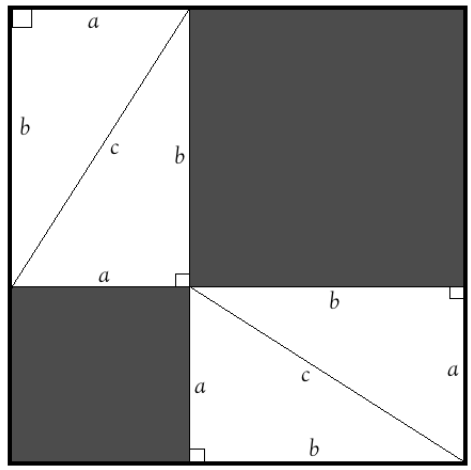

Figure 5: Rearranging the triangles 'shows' us the Pythagorean theorem

Notice that the total area of our figure did not change, because all we did was move the right triangles around. Then, since the area of the 4 right triangles has also not changed, this tells us that the area of our two shaded regions must equal the area of our original shaded region, c^2. But, the area of the smaller shaded region is just a^2, and the area of the larger shaded region is just b^2.

Therefore, we conclude that $a^2 + b^2 = c^2$.

Exterior Angles of A Convex Polygon

You probably know what a polygon is - a closed 2-D shape. The angles between the sides of the polygon (as measured from inside) are called the **interior angles** of the polygon. The angles made between a side and another adjacent *extended* side, is called an **exterior angle**. Figure 6 below shows a triangle and it's interior and exterior angles.

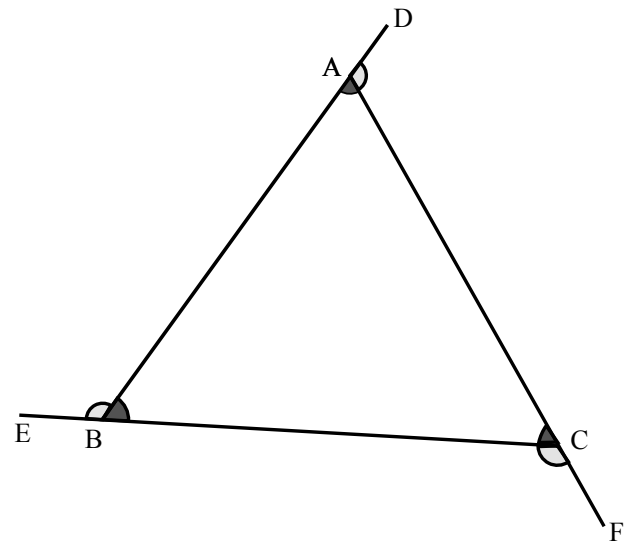

Figure 6: The interior and exterior angles of a triangle

Table 1: Interior and Exterior Angles

Interior Angles	Exterior Angles
$\angle BAC$	$\angle CAD$
$\angle ABC$	$\angle ABE$
$\angle BCA$	$\angle BCF$

There is a general theorem that talks about the interior angles of **any** polygon. It states that if you've got an $n - sided$ polygon:

$$Sum\ of\ interior\ angles = (n-2) \times 180^\circ$$

There's another (arguably more interesting) theorem about the exterior angles of any polygon. It states:

The sum of the exterior angles of any polygon is 360°

Now this is a pretty bold claim to make - that if you take any polygon at all and add up the exterior angles, you'll always end up with 360°. We're going to prove this theorem for a triangle (for simplicity) - the proof works the same way for any polygon with n sides.

Let's use our triangle from Figure 6. We need to prove that:

$$\angle CAD + \angle ABE + \angle BCF = 360^\circ$$

The first question is "Where to start?" Well, we know the sum of the interior angles of a triangle:

Sum of interior angles of triangle $ABC = \angle BAC + \angle ABC + \angle BCA = (3-2) \times 180^\circ = 180^\circ$

Now we need to somehow relate the interior angles to the exterior angles. To do this, notice that the interior angle $\angle BAC$ and the exterior angle $\angle CAD$ lie on a line. In other words, they form a **linear pair**. This means that corresponding pairs of interior and exterior angles must add up to 180°. So:

$$\angle BAC + \angle CAD = 180^\circ$$

$$\angle ABC + \angle ABE = 180^\circ$$

$$\angle BCA + \angle BCF = 180^\circ$$

Now we want to show that the sum of all the exterior angles is 360°. To do this, we can write each exterior angle in terms of it's corresponding interior angle, as shown below:

$$\angle CAD = 180^\circ - \angle BAC$$

$$\angle ABE = 180^\circ - \angle ABC$$

$$\angle BCF = 180^\circ - \angle BCA$$

Now we just add up the exterior angles:

$$\textit{Sum of exterior angles} = \angle CAD + \angle ABE + \angle BCF$$

$$\textit{Sum of exterior angles} = 180^\circ + 180^\circ + 180^\circ - (\angle BAC + \angle ABC + \angle BCA)$$

We can rewrite this to emphasize a point:

$$\textit{Sum of exterior angles} = 3 \times 180^\circ - (\textit{Sum of interior angles})$$

There's a term on the right hand side that's just the sum of the interior angles of our triangle! We already know that the sum of the interior angles in our triangle is 180°. This means that:

$$\textit{Sum of exterior angles} = 3 \times 180^\circ - 180^\circ = 2 \times 180^\circ = 360^\circ$$

This is what we wanted to prove in the first place, which means that we're done!

Now it's important to note that although we proved the exterior angle theorem for a triangle, it works for any polygon. In fact, if we did our proof for an $n - sided$ polygon, we would've proved the exterior angle theorem for **any polygon** - the method is exactly the same.

A Note About Proofs

Notice that in both proofs we did above, we started from what we know and showed that some other fact about the thing we're interested in was true. In the first case, we used the fact that the area of a square is simply the square of it's side to prove the Pythagorean theorem.

In the second case, we started with two facts: the sum of interior angles of a triangle and the relation between interior and exterior angles i.e. that they add up to 180°.

Each step in our proof was logically valid - we didn't have to assume anything was true. This method of proof is called a **direct proof**.

Compared to a direct proof, an **indirect proof** involves showing that every alternative to a fact leads to something weird (technically speaking - **an absurdity**). For example, an indirect proof of the exterior angle theorem would show that if the sum of the exterior angles of a polygon were not 360°, some well known fact about polygons wouldn't be true. Both kinds of proofs have their uses - some things are harder to prove one way than the other.

Shapes of Constant Width

We're going to take a break from proofs and look at a very quirky topic - shapes of constant width.

A shape of constant width is a 2-D figure with a special property - any line drawn through the center that connects two points on the outer edge of the shape has the same length.

You already know a shape like this - a circle. Any line drawn through the center of a circle that connects two points along its outer edge (i.e. a **diameter**) is the same length.

A famous shape of constant width is called the **Reuleaux triangle** (Figure 7), after Franz Reuleaux (1829 - 1905), a German engineer.

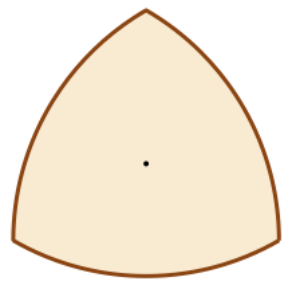

Figure 7: A Reuleaux Triangle

You might've seen some pencils with a cross-section that is a Reuleaux triangle. Some guitar picks are also really close to a Reuleaux triangle. Some other shapes of constant width are shown below in Figure 8. The ones shown are all **Reuleaux polygons**.

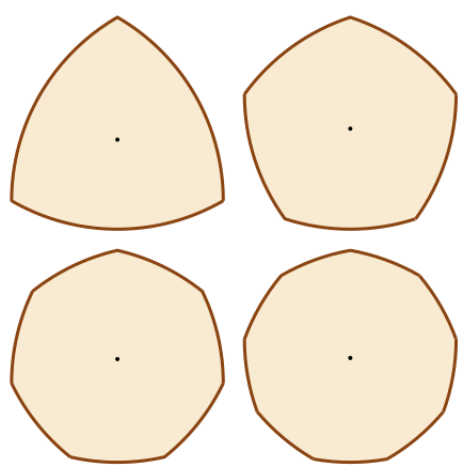

Figure 8: Other Shapes of Constant Width

To make your own Reuleaux polygon:

1. Start with a regular polygon with an odd number of sides.
2. Place a compass at one vertex of your polygon and draw a circle that passes through exactly 2 of the opposite vertices.
3. Repeat Step 2 for all vertices.
4. Erase the circles **except for the arcs connecting adjacent vertices**.

Example 1:

Make your own Reuleaux triangle by following the steps above (use the Figure below for reference).

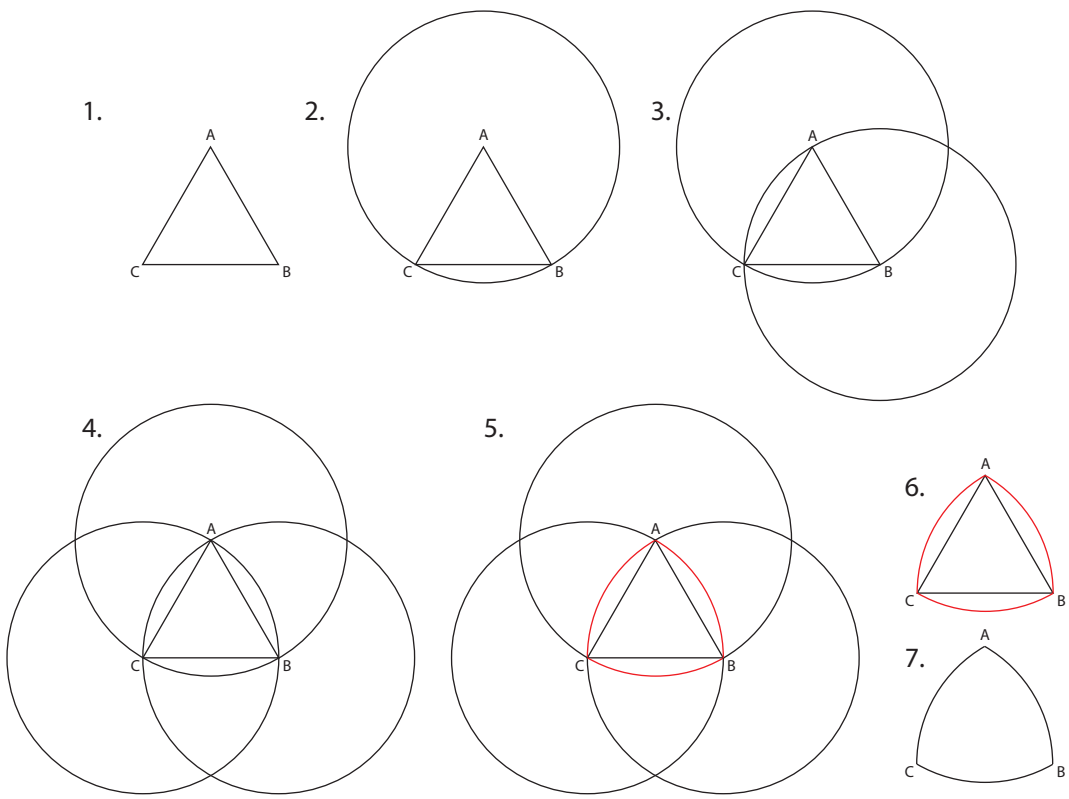

Figure 9: Make your own Reuleaux Triangle

To see another interesting thing about shapes of constant width, imagine making rollers out of them. Suppose you make 3 rollers that have a Reuleaux triangular cross-section. You place a board on top of your new rollers. Now, if you roll your board by pushing it along a curious thing happens - the height of the board above the table doesn't change (Figure 10).

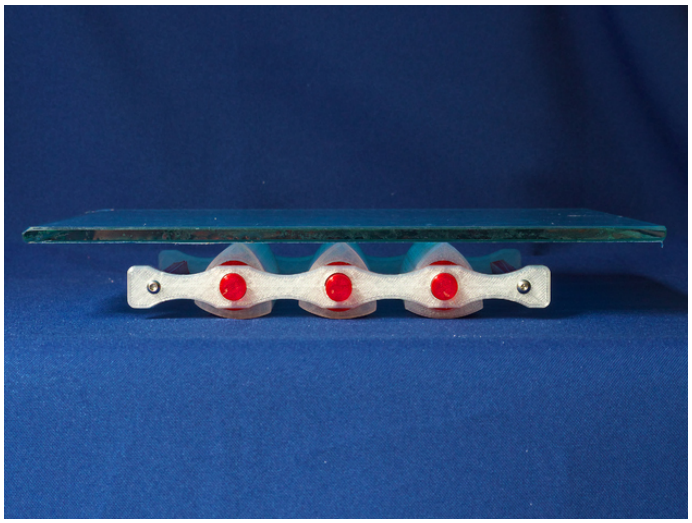

Figure 10: Rollers with Reuleaux Triangular Cross-Sections

The reason for this should be apparent to you - by placing a board on the rollers, you're measuring the height of the roller. The rollers are shapes of constant width so the height shouldn't change.

So since a Reuleaux triangle is a shape of constant width, we should be able to use it as a wheel right? Yes! In fact, a bicycle with Reuleaux triangle wheels was built in 2009.

So why don't we use wheels that are shaped like Reuleaux polygons? The primary issue has to do with the center of rotation. For example, if you look carefully at a Reuleaux triangle, you'll notice that the center of the triangle moves up and down as it rotates. Compare this to a circle whose center doesn't move when in rotates.

This means that the center of rotation of a Reuleaux polygon isn't fixed like it is for a circle. So any wheel made out of them would require clever designs at the axle to make them work.

Problem Set

1. Find the length of the hypotenuse of a triangle whose base is 4 units and height is 3 units. Is your answer a natural number?

2. Suppose a right triangle has a hypotenuse of length 7 units and a base of length 2 units. Find the height.

3. Is a right triangle with sides 12 cm, 7 cm and 6 cm possible? Why or why not? Explain.

4. A right triangle has it's base equal to it's height. How many times the base is the hypotenuse?

5. What is the sum of the interior angles of a heptagon (7 sides)?

6. The sum of interior angles of a polygon is 1440°. How many sides does the polygon have? If you were given the sum of exterior angles instead of the sum of interior angles, do you think you could've figured out the number of sides? Why or why not?

7. The geometry you're familiar with is called **Euclidean geometry**. It deals with flat spaces and falls apart when talking about curved spaces (like the surface of a sphere for instance). Albert Einstein worked with a special geometry called **Minkowski geometry** for his theory of relativity.

   The Pythagorean theorem is also affected by this weird geometry. It turns into:

   $$(Base)^2 - (Height)^2 = Hypotenuse^2$$

   Note the negative sign.

   Suppose you had an isosceles right triangle in this weird geometry, with base and height equal to 2 units. What is the length of the hypotenuse? Is your answer unexpected? Why or why not?

8. Construct a Reuleaux pentagon using nothing but a ruler, pencil and compass. Use the template given with this handout.

9. Consider the Reuleaux pentagon you made in Problem #8. Is the center of rotation of your pentagon fixed? Why or why not? Explain.