An Introduction to Graph Colouring

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Recap...

Last week, we covered:

- What is a graph?
- Eulerian circuits
- Hamiltonian Cycles
- Planarity

Reminder:

A graph G is:

- a set $V(G)$ of objects called **vertices**

together with:

- a set $E(G)$, of what we call called **edges**. An edge is an unordered pair of vertices.

We call two vertices **adjacent** if they are connected by an edge.

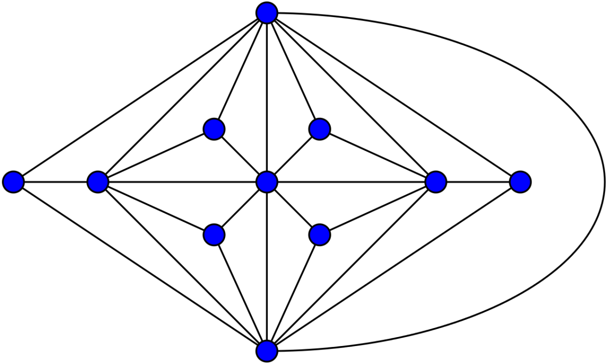
Today, we'll get into...

- Planarity in more detail
- The four colour theorem
- Vertex Colouring
- Edge Colouring

Recall...

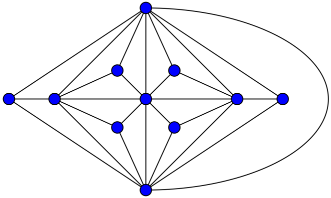
Planarity

We said last week that a graph is planar if it can be drawn in such a way that no edges cross.



Planarity

You'll notice the edges of planar graphs cut up our space into different sections.



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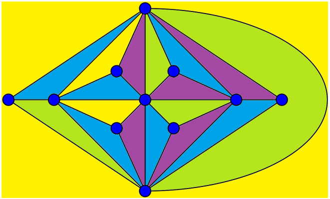

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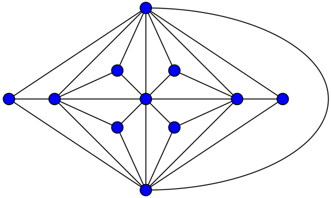

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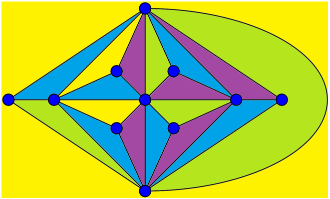
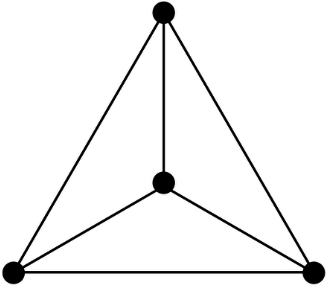


These areas, including the infinite area surrounding the graph, are called **faces**. We denote the set of faces of a graph G by $F(G)$.

Degree of a Vertex

Last week, we defined the degree of a vertex to be the number of edges that had that vertex as an endpoint.

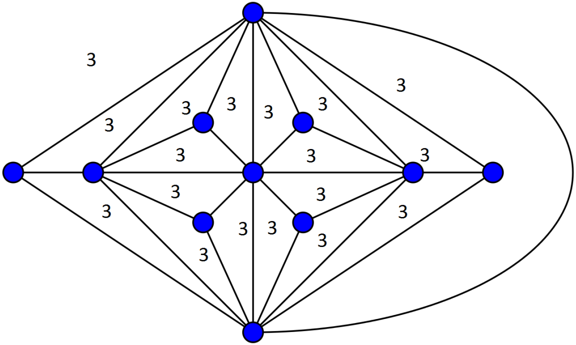


In this graph, for example, each of the vertices has degree 3.

Degree of a Face

Similarly, we can define the degree of a face. The degree of a face is the number of edges that make up the boundary of that face.

Take for instance the graph we saw earlier. The degree of each face is written in the face.



The Handshaking Lemma

Last week, we saw a useful equation called the Handshaking Lemma. This equation gave us a way of relating the edges and vertices of a graph.

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Handshaking Lemma

Let G be a graph.

$$2|E(G)| = \sum_{v \in V(G)} deg(v)$$

It would be helpful to come up with a similar expression relating the faces of a graph to its vertices or edges.

The Faceshaking Lemma

The Faceshaking Lemma

Let G be a graph.

$$2|E(G)| = \sum_{f \in F(G)} deg(f)$$

We can reason through this the same way we did the handshaking lemma – i.e. via a double counting argument.

Important observation: Each edge is in the boundary of exactly two faces.

Some Exercises

1. Draw a planar graph in which each vertex has degree 4 and each face has degree 3.

2. Draw a planar graph in which each vertex has degree 3 and each face has degree 5.

3. Draw a planar graph in which each vertex has degree 5 and each face has degree 3.

Euler's Formula

Euler's Formula is a handy way of relating the number of faces, vertices, and edges in a planar graph.

Euler's Formula

Let G be a planar graph.

$$|V(G)| - |E(G)| + |F(G)| = 2.$$

Why is this useful? We can use this to prove that planar graphs must have certain structural properties.

Structural properties of planar graphs

Theorem

Let G be a connected planar graph with $|V(G)| \geq 3$ vertices and $|E(G)|$ edges.

$$|E(G)| \leq 3|V(G)| - 6$$

Proof. As G has at least three vertices, each face in G has degree at least three. This means the average degree of the faces is at least three. This means that:

$$\begin{aligned}
2|E(G)| &= \sum_{f \in F(G)} deg(f) \\
2|E(G)| &= |F(G)|(\text{average degree of faces}) \\
2|E(G)| &\geq 3|F(G)|
\end{aligned}$$

Structural properties of planar graphs

Theorem

$$|E(G)| \leq 3|V(G)| - 6$$

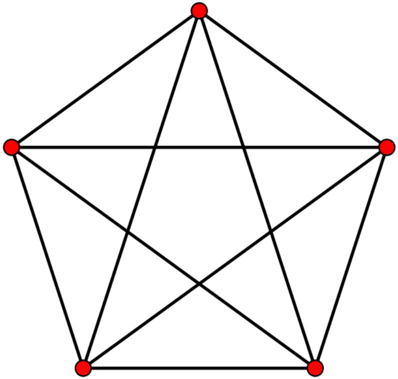
Proof (cont.)

$$\begin{aligned}
2|E(G)| &\geq 3|F(G)| \\
\text{By Euler's formula, } 2 &= |V(G)| - |E(G)| + |F(G)| \\
\text{Multiplying this by 3: } 6 &= 3|V(G)| - 3|E(G)| + 3|F(G)| \\
\text{And so: } 6 &\leq 3|V(G)| - 3|E(G)| + 2|E(G)| \\
6 &\leq 3|V(G)| - |E(G)| \\
\text{Rearranging: } |E(G)| &\leq 3|V(G)| - 6
\end{aligned}$$

... as required.

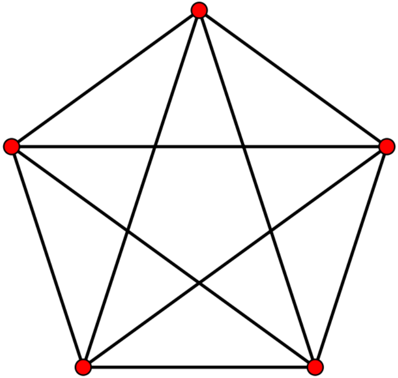
Exercise

Recall: K_5 is the complete graph on 5 vertices.



Exercise. Show K_5 isn't planar.

Solution



We know $|E(K_5)| = 10$, and $|V(K_5)| = 5$. By our previous theorem, if a graph G is planar, then $|E(G)| \leq 3|V(G)| - 6$. But for K_5, this would mean:

$$
\begin{aligned}
|E(G)| &\leq 3|V(G)| - 6 \\
|E(K_5)| &\leq 3|V(K_5)| - 6 \\
10 &\leq 3 \times 5 - 6 \\
10 &\leq 9
\end{aligned}
$$

... which is clearly false.

A Useful Corollary

Corollary

Every planar graph has a vertex of degree at most 5.

Proof. From before, we know that

$$|E(G)| \leq 3|V(G)| - 6 \tag{1}$$

$$\text{Multiplying by 2, we get } 2|E(G)| \leq 6|V(G)| - 12 \tag{2}$$

$$\text{Dividing by } |V(G)|,\ \frac{2|E(G)|}{|V(G)|} \leq 6 - \frac{12}{|V(G)|} \tag{3}$$

$$\text{And so... } \frac{2|E(G)|}{|V(G)|} < 6 \tag{4}$$

But by the handshaking lemma, $2|E(G)| = \sum_{v \in V(G)} deg(v)$. This means $\frac{2|E(G)|}{|V(G)|}$ is just $\frac{\sum_{v \in V(G)} deg(v)}{|V(G)|}$... which is just the average degree of the vertices.

This means line (4) is just saying the average degree is less than 6 – so there must be at least one vertex that has degree less than 6.

Onwards!

So far:

- We've taken a closer look at what it means for a graph to be planar.
- We've seen Euler's formula and the faceshaking lemma, and how this can be used to deduce structural properties of planar graphs.

Next:

- We'll take a look at the world of graph colouring.

Colouring Maps

Question: How many different colours do we need to colour a map in such a way as to ensure each adjacent region receives a different colour? (regions can share a colour if they meet at a corner, but not if they share a side)

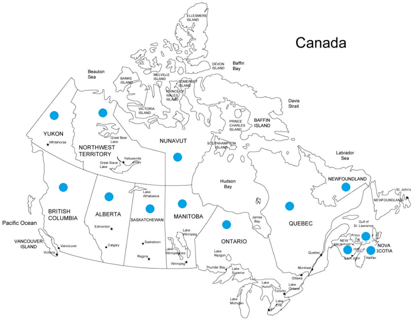
Colouring Maps

We can (surprise!) represent this as a problem on a graph. Draw a vertex in each province/territory...

$\Rightarrow$

Colouring Maps

Next, draw an edge between vertices if the corresponding provinces or territories border each other.



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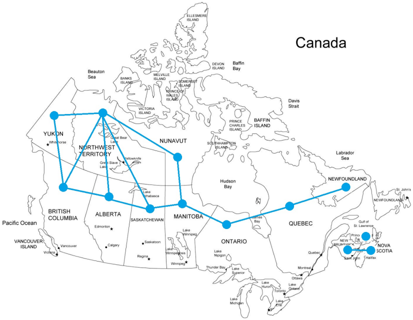
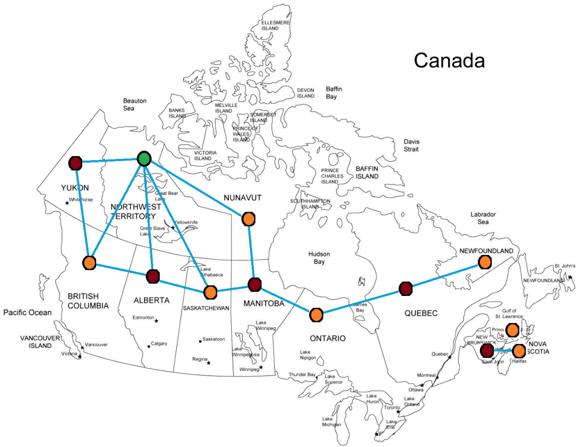


Colouring Vertices

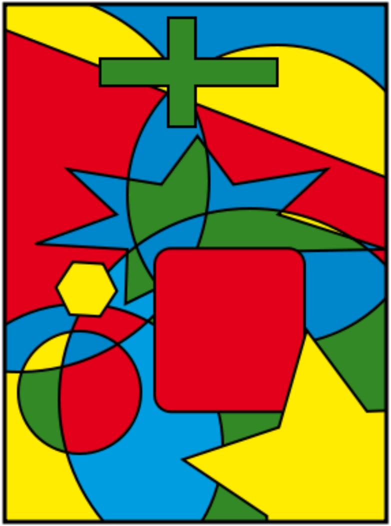
The question now becomes: how many colours do we need to colour the vertices of this graph in such a way that no two adjacent vertices are the same colour?



In the case of Canada, the answer is three.

Colouring Vertices

What about for more complicated maps?



In the case of more complicated maps like this one, we would need four colours.

Colouring Vertices

The Four Colour Theorem

Every planar map can be properly coloured using only four colours. (Or, in graph theory terms: the vertices of every planar graph can be properly[1] coloured using only four colours.)

This was first proven in 1976 by Kenneth Appel and Wolfgang Haken. It is one of the most famous theorems in graph theory, but for a long time, it was perhaps also the most controversial. Appel and Haken's proof was **computer-assisted**, which was a proof technique a lot of people didn't trust at the time.

[1] A **proper** colouring is one where no two adjacent vertices get the same colour.

The SixColour Theorem

Alright, so we won't be able to prove the four colour theorem at this point... but using what we've learned in this lecture, we *can* prove a weaker version.

The Six Colour Theorem

Every planar graph can be properly coloured using only 6 colours.

Proof. If the graph has fewer than 7 vertices, the result follows immediately. So for very small graphs it works. Let's assume now that there are bigger planar graphs that *can't* be coloured using six colours – let's call G the smallest possible counterexample[2]. Earlier, we showed every planar graph had a vertex of degree at most five. Call that vertex v.

[2] i.e. the counterexample with the smallest possible number of vertices.

The Six Colour Theorem

Let G' be the graph obtained from G by deleting v and all edges incident to it. Since G was the smallest counterexample, and G' has fewer vertices than G, this means G' is *not* a counterexample, so it can be coloured with six colours.

Now add v and all its edges back into G'. v was adjacent to at most five vertices in G, so it sees at most five colours in the colouring. This means there is at least one of the 6 colours remaining, so we can colour v.

We conclude G was not a counterexample after all.

What can we say about non-planar graphs?

First, a few definitions:

k-colourability

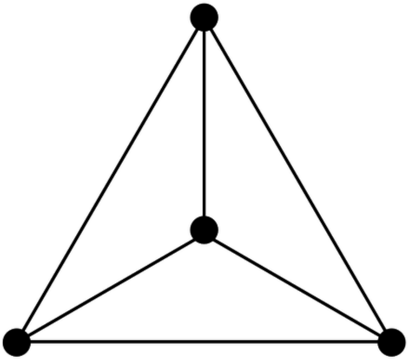
We say a graph G is k-colourable if its vertices can be coloured using at most k colours.

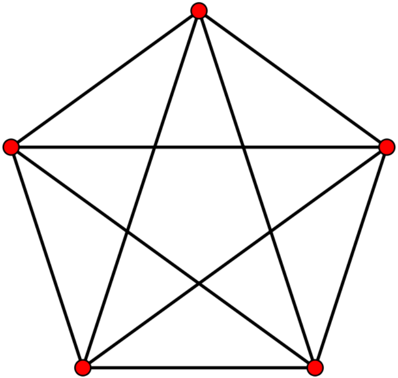
Chromatic number

The chromatic number of G, denoted $\chi(G)$, is the minimum k such that G is k-colourable.

Exercises.

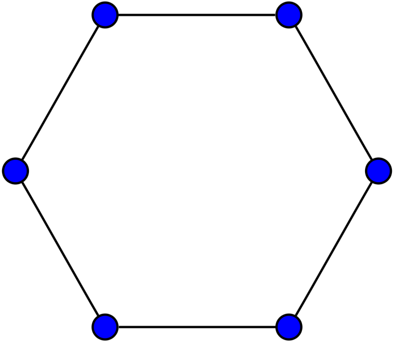
Find the chromatic number of each of the following graphs (K_4 and K_5).

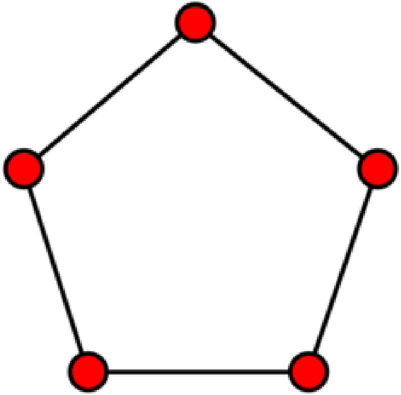




Exercises.

What is the chromatic number of an even-length cycle? What about an odd-length cycle?





Bounding $\chi(G)$

Brook's theorem gives an upper bound on the number of colours needed to properly colour the vertices of a graph.

Brook's Theorem

For any connected graph G with maximum degree Δ, the chromatic number of G is at most Δ unless G is a complete graph or an odd cycle – in which case the chromatic number is $\Delta + 1$.

Proof. This can be seen by colouring the vertices greedily.

Edge-colouring

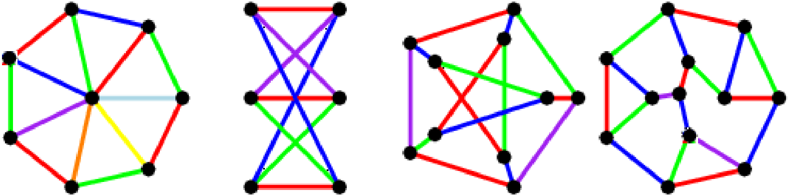
We've seen:

- What it means to vertex-colour a graph
- An upper bound for the chromatic number of planar graphs
- The chromatic number of complete graphs and cycles
- A trivial upper bound for the chromatic number of all graphs

Perhaps a natural question to ask now is: what about trying to colour the edges of a graph?

Edge-colouring

An edge-colouring of a graph G is a colour assignment to the edges of G where edges that share an endpoint get different colours.



Edge-colouring

We define the **chromatic index** of a graph G, written $\chi'(G)$, as the minimum number of colours required to properly colour the edges.

Can we find an upper bound for $\chi'(G)$?

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Can we find an upper bound for $\chi'(G)$?

Vizing's Theorem

For a graph G with maximum degree Δ, we have:

$$\chi'(G) \leq \Delta + 1.$$

Problem Set Time!

Next week...

- More problems on graphs (we'll revisit matchings)
- Ramsey Theory