

Matchings, Ramsey Theory, And Other Graph Fun

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Recap...

In the last two weeks, we've covered:

- ▶ What is a graph?
- ▶ Eulerian circuits
- ▶ Hamiltonian Cycles
- ▶ Planarity
- ▶ The Handshaking Lemma
- ▶ The Faceshaking Lemma
- ▶ Colouring Maps
- ▶ The Four Colour Theorem
- ▶ Vertex-Colouring
- ▶ Brook's Theorem
- ▶ Edge-Colouring
- ▶ Vizing's Theorem

Today, we'll get into...

- ▶ Matchings
- ▶ Covers
- ▶ Maximum and Stable Matchings
- ▶ Pigeonhole Principle Revisited
- ▶ Ramsey Theory (The Six Person Party Problem)

What is a matching?

Definition

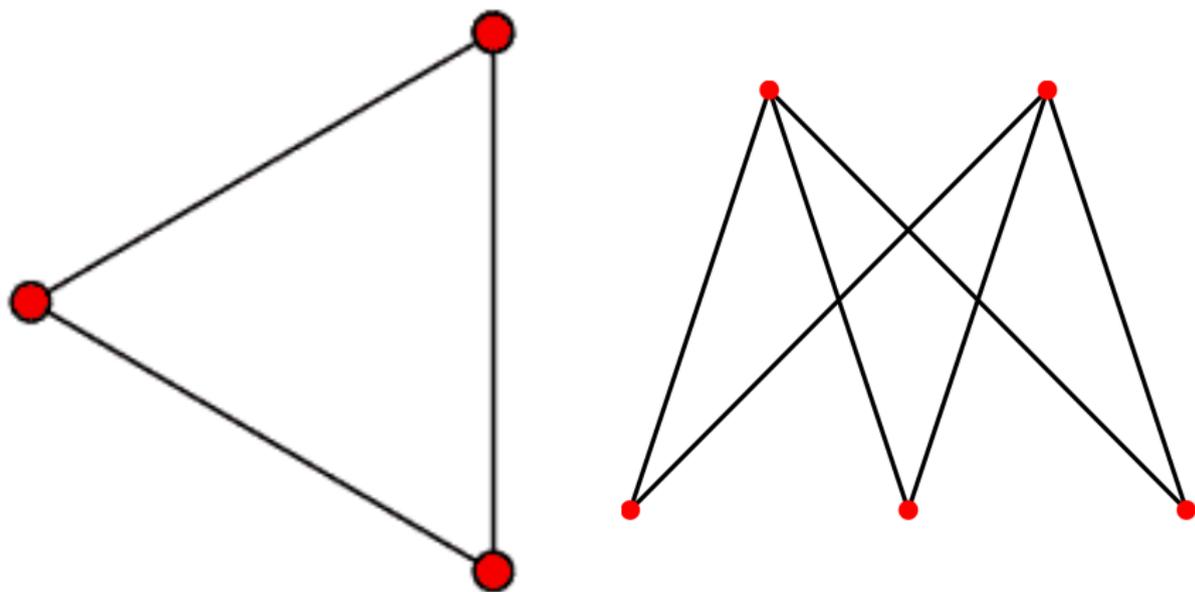
A **matching** M in a graph G is a set of edges such that no two edges share an endpoint. (So a matching matches certain pairs of adjacent vertices – hence the name.)

We say a vertex v is **saturated** by M if v is incident with an edge in M . (Otherwise, we say v is **unsaturated**.)

Every graph has a matching; the empty set of edges $\emptyset \subseteq E(G)$ is always a matching (albeit not a very interesting one).

Maximum Matching

The question we'll be most interested in answering is: given a graph G , what is the maximum possible sized matching we can construct?

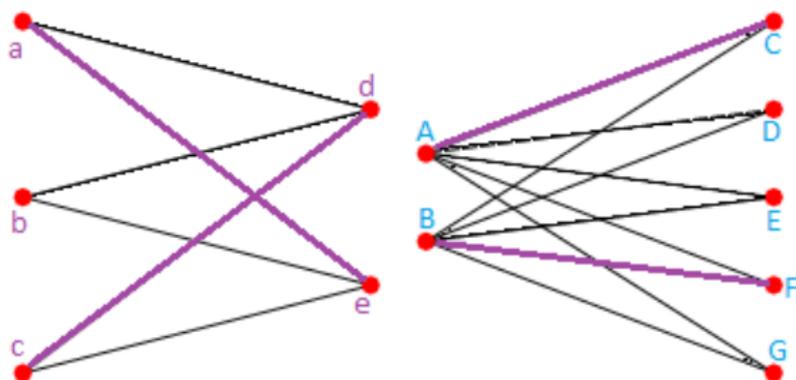


How can we tell if a matching is maximal?

Definition

An **alternating path** is a path with edges that alternate between being *inside* the matching and *outside* the matching.

In the following two graphs, the purple edges belong to matchings.

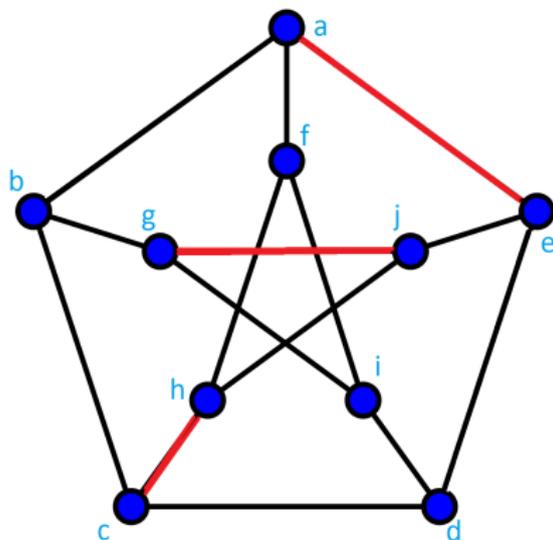


In the left graph, the path d, a, e, c is an alternating path. In the right graph, the path F, B, C, A, D is an alternating path.

How can we tell if a matching is maximal?

Definition

An **augmenting path** is an alternating path that starts and ends at an unsaturated vertex.¹

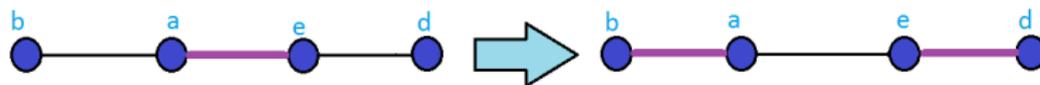


In this graph, the path b, a, e, d is an augmenting path.

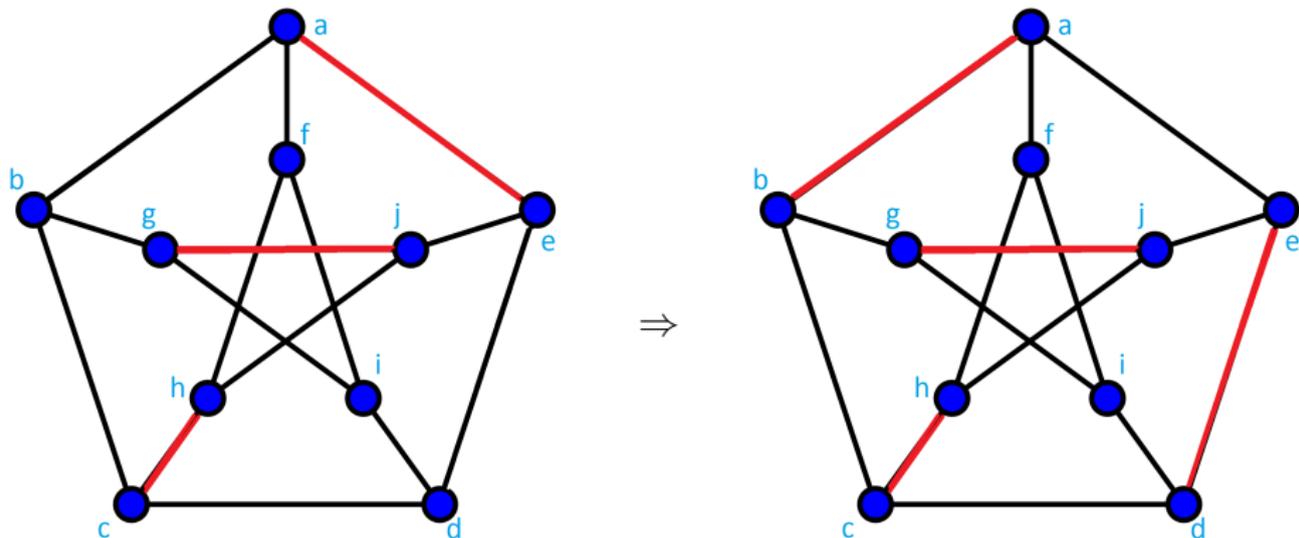
¹Recall, an unsaturated vertex is a vertex that isn't incident with an edge in a matching.

Why is it called an augmenting path?

We can use augmenting paths to augment the size of our matching. We do this by switching the matching and non-matching edges.



Why is it called an augmenting path?



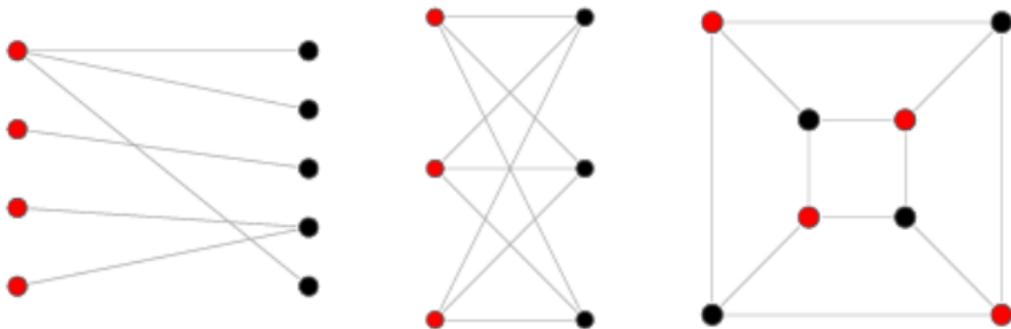
If there are no augmenting paths in our graph, we can conclude the matching is of maximum possible size. (Berge, 1957)

Bipartite Graphs

We will mostly be interested in matchings in bipartite graphs.

Definition

A **bipartite graph** is a graph whose vertices can be decomposed into two disjoint sets such that no two vertices within the same set are adjacent.



Matching Applications

The Job Assignment Problem

Say we have a set A of workers and a set B of jobs, and for each job, a list of workers capable of doing the job. We want to assign as many jobs to workers as possible, but each job can only be assigned to one worker, and each worker can only be assigned to one job.

We can model this problem as finding a maximum matching in a bipartite graph.

How else can we tell if our matching is maximal?

We call a set $C \subseteq V(G)$ a **cover** of G if every edge has at least one endpoint in C . It's always easy to find large covers, just as it is always easy to find small matchings.

Theorem

If M is a matching of G and C is a cover of G , then $|M| \leq |C|$.

Proof. For each edge $\{u, v\}$ in M , either u or v is in C . Moreover, for any two edges of M , the vertices of C they saturate must be different, since M is a matching. Therefore for each edge in M there is at least one vertex in C .

How else can we tell if our matching is maximal?

We can sometimes use covers to tell whether a matching is of maximal size.

Lemma

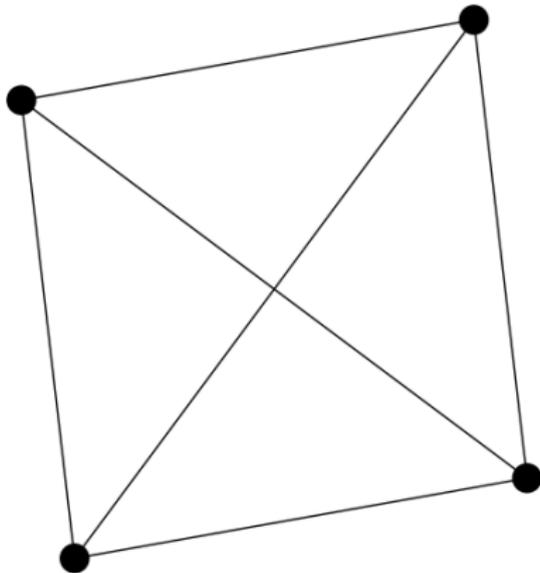
If M is a matching, C is a cover and $|M| = |C|$, then M is a maximum matching and C is a minimum cover.

Proof. Let M' be any matching. Then from our theorem earlier, $|M'| \leq |C| = |M|$. It follows then that M is a maximum matching. Now let C' be any cover. Again, from the earlier theorem, we have $|C'| \geq |M| = |C|$. It follows that C is a minimum cover.

Exercise.

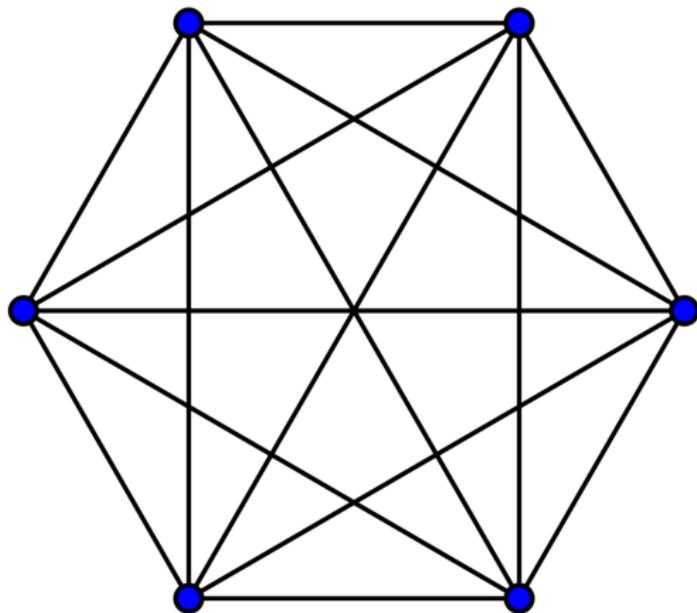
We call a matching **perfect** if all of the vertices are saturated by the matching.

K_4 is the complete graph on four vertices. How many different perfect matchings are there in K_4 ?



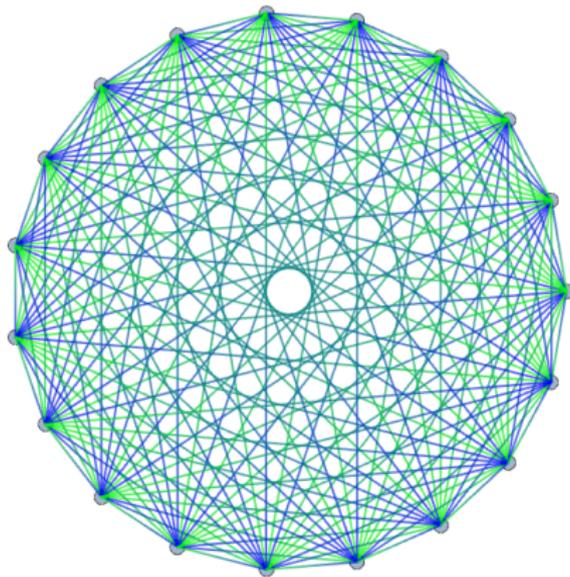
Exercise.

How many perfect matchings are there in K_6 ?



Exercise.

How many perfect matchings are there in K_{19} ?



Stable matchings

The stable matching problem is the problem of finding a stable matching between two equally sized sets of elements given an ordering of preferences for each element. A matching is **stable** if there are no two vertices that would rather be matched to each other than their current partners.

The stable matching problem has been stated as follows:

Given n men and n women, where each person has ranked all members of the opposite sex in order of preference, find a way to marry the men and women together such that there are no two people of opposite sex who would both rather have each other than their current partners.

Stable matchings

We can model this as finding a stable matching in a complete bipartite graph. One side of the bipartition represents the men, and the other side, the women; when two people become engaged, we add the edge between them to the matching.

Stable matchings

The **Gale-Shapley algorithm** for stable matchings gives us a way to find a stable matching in a complete bipartite graph.

The algorithm goes as follows. In the first round:

- ▶ Each unengaged man proposes to the woman he prefers most
- ▶ Each woman answers **maybe** to her suitor she most prefers and **no** to all other suitors.

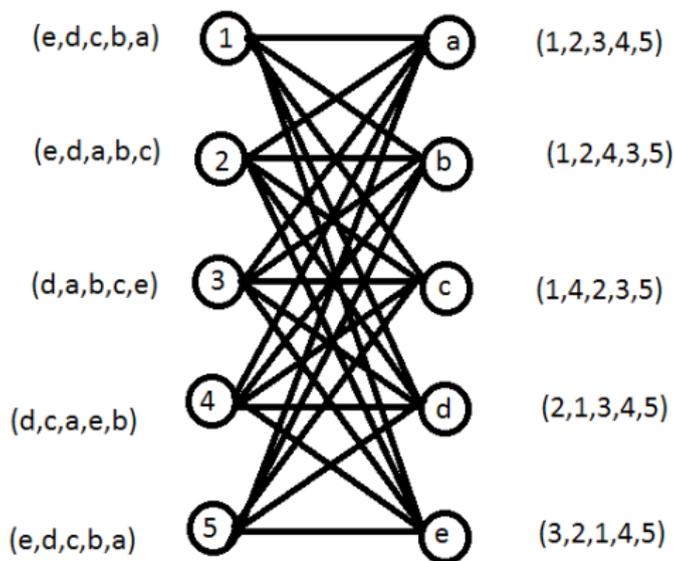
She is then engaged to the suitor, and that suitor is likewise engaged to her. In each subsequent round...

- ▶ Each unengaged man proposes to the most-preferred woman to whom he has not yet proposed (regardless of whether the woman is already engaged)
- ▶ Each woman replies **maybe** if she is currently not engaged or if she prefers this guy over her current provisional partner (in this case, she rejects her current fiancé who then becomes single).

This process is repeated until everyone is engaged.

Exercise.

Run the Gale-Shapley algorithm to find a stable matching in the following graph (preference lists are next to the vertices).



The Pigeonhole Principle

If $n + 1$ pigeons roost in n pigeonholes, then at least one pigeonhole will have more than one pigeon in it.



More generally: say you have m objects that you need to sort into n categories. At least one category will have at least $\lceil \frac{m}{n} \rceil$ objects in it. (In other words: not every category can have fewer than the average number of objects in it.)

Ramsey Theory

Ramsey theory is a subfield of mathematics named after the British mathematician Frank P. Ramsey. It studies the conditions under which some sort of order necessarily appears. Questions in Ramsey theory are typically of the form “how big must some structure be to guarantee that a particular property will hold?”

We can rephrase the pigeonhole problem this way: if we have n pigeonholes, how many pigeons must come to roost to guarantee at least one pigeonhole has more than one pigeon in it?

The Six-Person Party Problem

For the following few slides, we'll be working in a highly idealized world where everyone is either friends or strangers.

Claim: in a group of six people, there must be at least three mutual friends or three mutual strangers.

Of course, we can model this problem with a graph: draw a vertex for each person; draw a green edge between people if they are friends, and a blue edge between people if they are strangers. Three mutual friends are then a green triangle; three mutual strangers, a blue triangle.

The Six-Person Party Problem

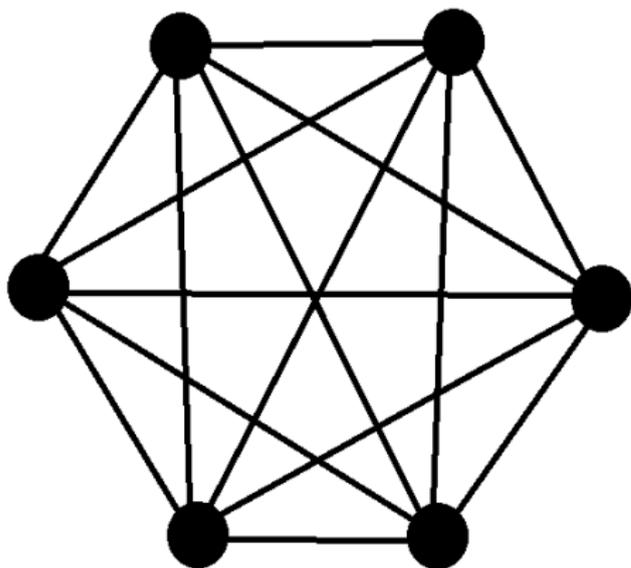
In a group of six people,
there must be at least
three mutual friends or
three mutual strangers.

⇒

No matter how you colour
the edges of K_6 using only
two colours, you are
guaranteed to have a
monochromatic triangle.

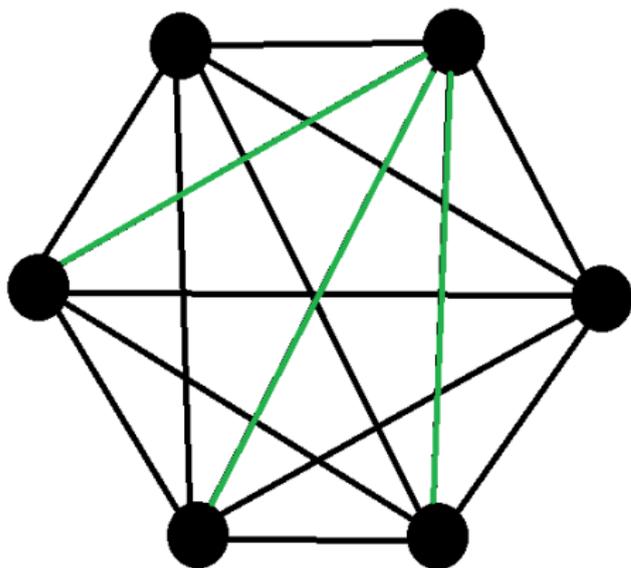
The Six-Person Party Problem

We begin with a graph on six vertices; none of the edges have yet been coloured. Consider now a single vertex v in the graph. This vertex is adjacent to five others; each of these five other vertices fit into one of two categories: friends of v , or strangers to v .



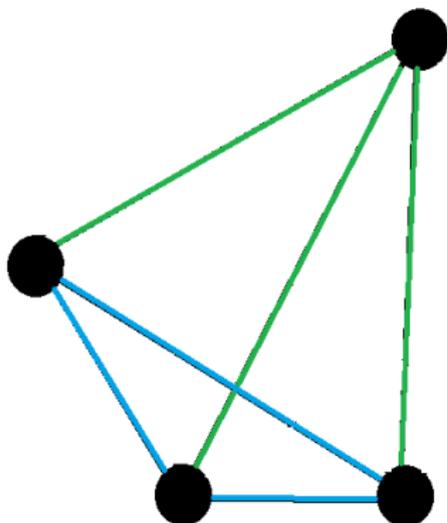
The Six-Person Party Problem

We have five objects (the vertices adjacent to v) and two categories (friends of v or strangers to v). One of the categories must have ≥ 3 objects in it – i.e. v has either ≥ 3 friends or ≥ 3 strangers amongst its neighbours.



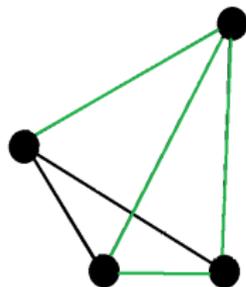
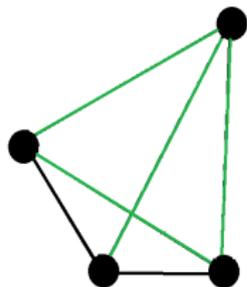
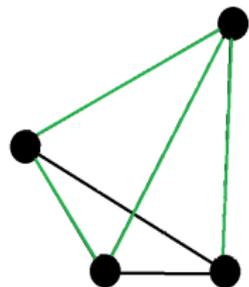
The Six-Person Party Problem

Let's restrict our analysis to only v and these three neighbours. If the three neighbours are strangers, we've found a monochromatic blue triangle, and so we are done.



The Six-Person Party Problem

If even one of the three neighbours is friends with v , we have a monochromatic green triangle.



The Six-Person Party Problem

Would this be true with fewer people? In a group of five people, for instance, are you always guaranteed three mutual friends or strangers?

The Six-Person Party Problem

Would this be true with fewer people? In a group of five people, for instance, are you always guaranteed three mutual friends or strangers?

As it turns out: nope! Six is the minimum number of people to guarantee that this will always happen.

Definition

We call a **clique** in a graph a subset of the vertices that form a complete graph.

Definition

We call an **independent set** in a graph a subset of the vertices that do not have any edges between them.

Ramsey Numbers

Ramsey Numbers

The Ramsey number $R(r, s)$ is the minimum number of vertices you need in a graph in order to ensure you will have either a clique of size r , or an independent set of size s .

This is equivalent to our notion earlier of using two colours to colour the edges: think of cliques as sets of vertices all joined up by say, green edges; think of independent sets as vertices all joined up by blue edges.

Ramsey Numbers

Despite being first thought of nearly a century ago, not many Ramsey numbers are known.

r, s	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7	8	9	10
3	1	3	6	9	14	18	23	28	36	40–42
4	1	4	9	18	25	36–41	49–61	59 ^[8] –84	73–115	92–149
5	1	5	14	25	43–48 ^[9]	58–87	80–143	101–216	133 ^[9] –316	149 ^[8] –442
6	1	6	18	36–41	58–87	102–165	115 ^[8] –298	134 ^[8] –495	183 ^[9] –780	204 ^[9] –1171
7	1	7	23	49–61	80–143	115 ^[8] –298	205–540	217–1031	252 ^[9] –1713	292 ^[9] –2826
8	1	8	28	59 ^[8] –84	101–216	134 ^[8] –495	217–1031	282–1870	329 ^[9] –3583	343 ^[9] –6090
9	1	9	36	73–115	133 ^[9] –316	183 ^[9] –780	252 ^[9] –1713	329 ^[9] –3583	565–6588	581–12677
10	1	10	40–42	92–149	149 ^[8] –442	204 ^[9] –1171	292 ^[9] –2826	343 ^[9] –6090	581–12677	798–23556

Lower bounds for $R(6, 6)$ and $R(8, 8)$ have not been improved since 1965 and 1972, respectively. The upper bound for $R(5, 5)$ was very recently (March 26th) brought down from 49 to 48.

Ramsey Numbers

“Erdős asks us to imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of $R(5, 5)$ or they will destroy our planet. In that case, he claims, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for $R(6, 6)$. In that case, he believes, we should attempt to destroy the aliens.”

— Joel Spencer (American Mathematician)

Ramsey Numbers

Some results are fairly easy to prove, however:

$$R(1, k) = R(k, 1) = 1$$

$$R(2, k) = R(k, 2) = k$$

We could also show, not too painfully, that:

$$R(k, t) \leq R(k, t - 1) + R(k - 1, t)$$

Lower bounds are generally found by suitable graph constructions.

Back to colouring!

Recall last week we said every even cycle can be 2-coloured; i.e. we can properly² colour the vertices of every even cycle using only two colours.

Last week, we briefly spoke about list-colouring, which was just like normal colouring except we could restrict the colours available to each vertex using a list. For a graph G , the **list-chromatic number**, $\chi_l(G)$, is the minimum list-size needed to ensure G has a proper list-colouring, no matter what the lists look like.

Today, we will show every even cycle can be 2-list coloured. This means both the chromatic number and the list-chromatic number of even cycles are 2.

²Recall we call a vertex-colouring *proper* if no two adjacent vertices are the same colour.

Back to colouring!

Last week, we briefly spoke about **correspondence colouring**. We can think of correspondence colouring as list-colouring, except with more complicated rules. In list-colouring, like in traditional colouring, if a vertex was coloured red, none of its neighbours could be red (the rule was always: if vertex v is colour c , none of v 's neighbours can be c .)

Correspondence colouring is a bit different. For instance, we could say that for a vertex v with neighbour u , if v is coloured red, then that means u can't be coloured blue.

The correspondence chromatic number of a graph G is the minimum number of colours in each of the vertices' lists to ensure that the graph can be coloured, no matter what colouring rules look like. Today, we will show the correspondence chromatic number of even graphs – unlike the normal chromatic number and the list-chromatic number – is *not* 2.

Problem Set Time!

Questions?