UNIVERSITY OF WATERLOO

Faculty of Mathematics
Waterloo, Ontario N2L 3G1

Centre for Education in
Mathematics and Computing

Grade 7/8 Math Circles

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Number Theory

Introduction

Today, we will be looking at some properties of numbers known as **number theory**. Number theory is part of a branch of mathematics called *pure mathematics*. More specifically, we will learn about palindromes and triangular numbers, before looking at prime numbers and some other pretty neat stuff.

Palindromic Numbers

What do you notice about the following images?



These images are ________________. However, it is not only pictures that can follow this property: words and numbers can as well.

Palindrome: A word or phrase that reads the same forwards and backwards. For example, Madam, Hannah, and Go dog are palindromes.

Palindromic Number: A number that reads the same forwards and backwards. For example, 1331, 404, 9, 77777, and 145686541 are palindromic numbers.

Examples At the Waterloo Marathon, everyone has a bib with a number on it. You are watching the runners going by and taking note of their bib number.

(a) James is the smallest 3-digit palindromic number. What number is James?

(b) The product of Maureen's two digits is 49. What palindromic number is Maureen?

Finding Palindromic Numbers

One way to find a palindrome is as follows:

1. Pick any number
2. Reverse the digits of the number
3. Add these two numbers together
4. Repeat until you get a palindrome

Example Using the number 37, find a palindromic number.

Perfect Square Palindromic Numbers

Evaluate the following:

$11^2 =$

$101^2 =$

$1001^2 =$

$10001^2 =$

What is the pattern of these perfect squares?

Examples

(a) What is $1\ 000\ 000\ 000\ 000\ 000\ 001^2$?

(b) What is $\sqrt{1\,000\,000\,002\,000\,000\,001}$?

Perfect Cube Palindromic Numbers

Let's see if we can find a similar pattern with perfect cubes.

$11^3 =$

$101^3 =$

$1001^3 =$

$10001^3 =$

What is the pattern of these perfect cubes?

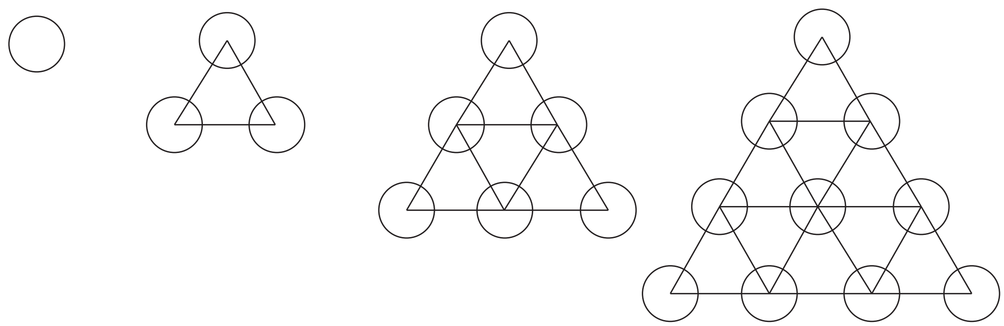
Examples

(a) What is $1\ 000\ 001^3$?

(b) What is $\sqrt[3]{1\,000\,000\,003\,000\,000\,003\,000\,000\,001}$?

Triangular Numbers

Consider the following pattern:



What is the rule of the pattern?

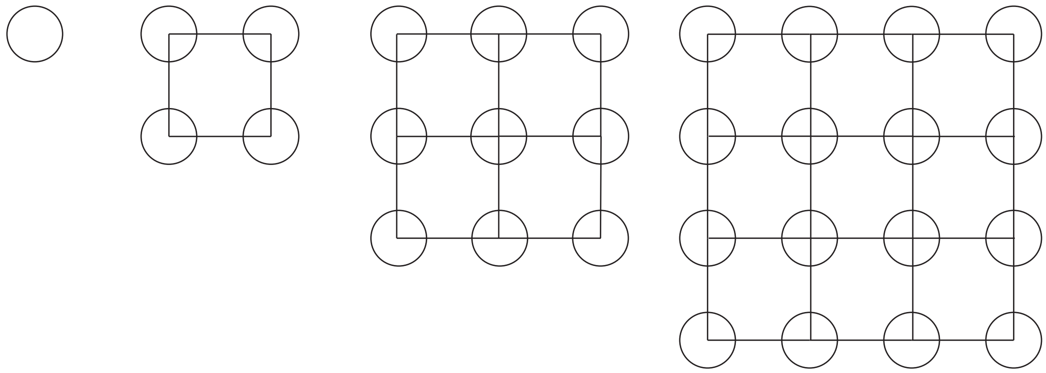
The number of dots in each triangle form a sequence of numbers that we call the **triangular number sequence**. The sequence of triangular numbers is as follows: {1,3,6,10,...}. We also see this sequence in Pascal's triangle!

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

From this sequence, we can find that the formula for triangular numbers is $\mathrm{t}_n = \dfrac{n(n+1)}{2}$, where t_n is the n^{th} term or the n^{th} triangle.

But triangles aren't the only shape we can consider. In fact, we can consider any shape, so we could have any set of **polygonal numbers**.

Let's consider the set of square numbers:



Notice that these numbers look familiar! We tend to know the square numbers since we know our square roots so well. We also call the set of square numbers **perfect squares** since their square roots are integers.

Examples

(a) What is the 13^{th} triangular number?

(b) What is the 13^{th} square number?

The Locker Problem

One hundred students are assigned lockers 1 to 100. The student assigned to locker 1 opens every locker. The student assigned to locker 2 then closes every other locker. The student assigned to locker 3 changes the status of all lockers whose numbers are multiples of 3 (If a locker that is a multiple of 3 is open, the student closes it. If it is closed, the student opens it). The student assigned to locker 4 changes the status of all lockers whose numbers are multiples of 4, and so on for all 100 lockers.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

You may use the grid above to help solve the following questions:

1. Which lockers are left open? And why were they left open?

2. Which lockers were touched exactly two times?

3. How do you know that these lockers were touched exactly two times?

Prime Numbers

A **prime** number is a natural number that can only be divided by 1 and itself.

A **composite** number is a natural number that has more **factors** than just 1 and itself.

For example, 2, 3, and 5 are prime numbers, since $2 = 2 \times 1$ but there is no other way to multiply to get 2. 4, 6, and 9 are composite numbers since $4 = 2 \times 2$ and $4 = 4 \times 1$.

One method for finding the prime numbers is by using the **Sieve of Eratosthenes**. Here are the steps to this algorithm, using the following table:

1. Cross out 1 (it is not prime)
2. Circle 2 (it is prime) and then cross out all multiples of 2
3. Circle 3 (it is prime) and then cross out all multiples of 3
4. Circle 5, then cross out all multiples of 5
5. Circle 7, then cross out all multiples of 7
6. Continue by circling the next number not crossed out, then cross out all of its multiples

The circled numbers are all the prime numbers less than 100.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Prime Factorization: Extended

Review: Find the prime factorization of 1440.

Fermat's Factorization Method

Find the prime factorization of 989.

We know that every positive number can be written as a product of prime factors and this product can be found using prime factorization. What happens if the prime factorization is composed of large prime numbers? It becomes difficult and to manually check if each prime number is a factor until one is found. Another method to help us find the prime factorization of such a number is something called **Fermat's Factorization Method**. Before we begin, we need to learn one new thing called the **difference of squares**.

Difference of Squares

Let a and b any number. Then,

$$a^2 - b^2 = (a+b)(a-b)$$

Examples Evaluate the following:

(a) $5^2 - 4^2$

(b) $12^2 - 5^2$

Now, to find the prime factorization of a number using Fermat's Factorization Method, we will use the following steps.

Suppose we want to find the prime factorization of the number n.

1. Choose the smallest number a such that $a^2 > n$.
2. Evaluate $a^2 - n$. If $a^2 - n$ is NOT a perfect square, then repeat step 2 with $(a+1)$, $(a+2), (a+3), \ldots$ until we find a perfect square.
3. Suppose a gives us a perfect square, b^2, in step 2. Then,

$$a^2 - n = b^2 \Rightarrow n = a^2 - b^2 = (a+b)(a-b)$$

4. If $(a+b)$ and $(a-b)$ are prime numbers, then we are done. Otherwise, we can do one of two things:
   (a) Repeat Fermat's Factorization Method with $(a+b)$ and/or $(a-b)$, OR
   (b) Use a factor tree to find the prime factorizations of $(a+b)$ and/or $(a-b)$.

Example Find the prime factorization of 1173.

Multiplicity of Prime Factors

Multiplicity: The number of times a prime factor is multiplied. For example, $9 = 3^2$. The multiplicity of 3 is 2.

Find the prime factorization of the following perfect squares:

(a) 25 (b) 81 (c) 144

What do you notice about the multiplicity of each prime factor in the examples above?

Interesting. Let's try to find the multiplicity of a few perfect cubes.

(a) 8 (b) 64 (c) 8000

What do you notice about the multiplicity of each prime factor in the examples above?

Therefore, the multiplicity of each prime factor of an n^{th} root is a multiple of n.

Examples

(a) If you know that 243 is a 5^{th} root, find its prime factorization.

(b) What is the multiplicity of the prime factors of 2187?

How Many Factors?

Without listing all the factors, how many factors does 1440 have?

We can answer this question using prime factorization! Consider this:

What is the prime factorization for each of the following numbers? List the factors for each number as well.

8

9

16

What do you notice?

What is the prime factorization of 24? List all its factors.

Number of Factors of N

Let N be any positive integer. Suppose the prime factorization of N is

$$N = 2^a \times 3^b \times 5^c \times \dots$$

where a, b and c are also positive integers. Then, the number of factors of N is

$$(a+1) \times (b+1) \times (c+1) \times \dots$$

More Fun With Primes!

In keeping with the topic of palindromes, a **palindromic prime** is a prime number that reads the same forwards and backwards. Here is a list of some palindromic primes:

2, 3, 5, 7, 11, 101, 131, 151, 181, 191, 313, 353, ...

Twin primes are pairs of primes numbers that are either two less or two more than each other. For example, (5,7), (17,19) and (41,43) are pairs of twin primes.

An **emirp** is a prime that gives you a *different* prime when its digits are reversed. For example, `13` becomes `31` when we reverse its digits and they are different numbers and they are both prime. A list of some emirp numbers are shown below:

13, 17, 31, 37, 71, 73, 79, 97, 107, 113, 149, 157, ...

A **Mersenne prime** is a prime number that is one less than a power of two. To be more clear, suppose `p` can by any whole number. Mersenne primes can be written as follows:

$$2^p - 1$$

The first three Mersenne primes are:

$$\begin{array}{ccc} 3, & 7, & 31 \\ \Downarrow & \Downarrow & \Downarrow \\ 2^2 - 1 & 2^3 - 1 & 2^5 - 1 \end{array}$$

Fun fact!
Currently, the largest known Mersenne prime is $2^{74207281} - 1$. It is 22 338 618 digits long!

Problem Set

1. How many 1-digit palindromic numbers are there?
2. Which of the following are palindromes?
   (a) HANNAHBANANABHANNAH
   (b) 1030201
   (c) ABCDECBA
   (d) 1771
3. Find a palindromic number from the following:
   (a) 18
   (b) 886
   (c) 3742
4. 2002 was the last palindromic year. What will be the next palindromic year?
5. How many 7-digit palindromic numbers are there?
6. How many palindromic numbers are there between the numbers 100 and 400?
7. Maria is finishing a marathon and Paul is waiting for her at the finish line. He cannot remember her bib number, just that it was the next palindromic number after his bib number, 5678. What bib number does Maria have?
8. (a) Evaluate the following:
      i. $\sqrt{100\,000\,020\,000\,001}$
      ii. $100\ 000\ 001^3$
   (b) (Square of Perfect Fourths) $101^4 = 104060401$. Evaluate the following:
      i. $\sqrt[4]{10\,004\,000\,600\,040\,001}$
      ii. 100001^4
   (c) What do you notice about the non-zero digits of the perfect square, cube, and fourth palindromic numbers?

9. Sachin plays for the Waterloo quidditch team and made sure he picked out a jersey number that is a palindrome. Quidditch rules state that his jersey number must be less than 1000. Sachin chose an even number, where the product of the digits in his number is 12. What is Sachin's jersey number?

10. (a) Find the prime factorization of 10 500.

    (b) How many factors does 10 500 have?

11. Given the expression $2^{2k}\ 3^{3k}\ 5^{5k} = 337\ 500$, what is k?

12. Find the prime factorization of the following numbers.

    (a) 4897

    (b) 1219

    (c) 4085

13. What is the multiplicity of the prime factors of 9 765 625?

14. Is 129 a Mersenne prime?

15. * **Locker Problem Extended** Which locker(s), from 1 to 100, were open and closed the most times?

16. ** An old cat lady meets with her fellow animal loving friend. The cat lady asks her friend, "How many dogs do you have?" The friend answers, "I have three dog whom I love very much." The cat lady then asks, "How old are your dogs?" The friend decides to test the cat lady's math skills and replies, "The product of their ages 288 and the sum of their ages is the day of month on which your birthday falls." After some thinking, the cat lady says, "Nope, I need more information." "Alright," says the friend, "The youngest dog still needs to be potty-trained." Immediately, the cat lady says, "Got it! I know their ages!!" How old are the dogs?

17. *** How integers n are there where $1 \leq n \leq 100$ and n^n is a perfect square?