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Grade 6 Math Circles

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Game Theory

Let's Play a Game!

Fred and George are sent to Professor Umbridge's office for setting off fireworks in the Great Hall. Professor Umbridge wants to give them detention but needs to get a confession out of them first.

She separates them and tells each one that:

- If neither of them confess, they will each get 1 detention
- If they both confess, they will each get 3 detentions
- If one confesses and the other denies, then the person who confesses will get no detentions while the other person gets 7!

Now, as a class, we will make a decision for each person: to Confess or Deny.

Player	Decision
Fred	
George	

Game Theory

Game Theory studies the mathematics and logic behind how players make decisions while playing games. In Game Theory, we study **Mathematical Games** which have rules, no cheating, and two or more players whose decisions and outcomes are dependent on the other players. The concepts and strategies we will learn today are not only applicable to many games but also to real life situations. Game Theory is used in studying economics, political science, computer science, and biology.

Simultaneous Games

In **simultaneous games**, you choose your move without knowing what other players are going to do. Everyone's decision is revealed at the same time.

Example: Prisoner's Dilemma

The game we played at the beginning of class is called Prisoner's Dilemma and it is a type of simultaneous game. It involves 2 players, each having to choose a strategy of Confess or Deny without knowing what the other player does.

What is the **goal** of this game? What is **winning**?

To solve this problem, we visualize it with a **payoff table**. The numbers in this table are called **payoffs** which represent a player's outcome (usually the amount of their profit or loss) for each possible situation.

In our example, the payoffs we look at are the number of detentions each person would get for each of the 4 situations that can occur. In each section of the payoff table, payoffs are represented as "a, b" where a is the payoff for the player at the left of the table (Fred) and "b" is the payoff for the player on the top of the table (George).

		GEORGE	
		Confess	Deny
FRED	Confess	3, 3	
	Deny		

Now that the values of the payoff table has been filled in, would you change your vote?

Next, we put ourselves into Fred's shoes and decide what is the best move to make depending on what George's decision might be:

- If George confesses, Fred would get 3 detentions if he confesses and 7 if he denies
- If George denies, Fred would get 0 detentions if he confesses and 1 if he denies

So in both cases, Fred should confess. Let's mark this on our table.

		GEORGE	
		Confess	Deny
FRED	Confess	* 3, 3	* 0, 7
	Deny	7, 0	1, 1

Let's do the same for George and think what he should do depending on Fred's decision:

- If Fred confesses, George would get 3 detentions if he confesses and 7 if he denies
- If Fred denies, George would get 0 detentions if he confesses and 1 if he denies

Again, in both cases George's best option is to confess. It makes sense that his best strategy would be the same as Fred's because they were given the exact same offer.

		GEORGE	
		Confess	Deny
FRED	Confess	* **3, 3** *	* 0, 7
	Deny	7, 0 *	1, 1

We see that the best strategy for both Fred and George is that they both confess which results in 3 detentions each. How does this result compare with our choice at the beginning of class?

But wait! Why would both Confessing be Fred and George's best option when it would give them both 3 detentions while both Denying would give them each only 1 detention. Let's suppose Fred realized this and decided to change his answer to Deny. Now, George can also change his answer to Deny so they will each get only 1 detention *or* he could keep his answer at Confess so that he gets no detentions at all! This would mean that Fred gets 7 detentions total! Not wanting to risk having so many detentions, Fred decides that he should Confess after all. This is why, even though both Denying would result in 2 less detentions for each person, both should decide to Confess or else they risk getting 7 detentions.

This shows how a player's strategy changes when considering what an opponent might do.

Nash Equilibrium

In the previous example, we highlighted the situation that resulted from the both players using their best strategy. This is called a **Nash Equilibrium**.

A Nash Equilibrium is the state where each player has chosen the strategy with the best payoff for themselves, while considering other players' decisions. This means thinking about not only one player's best strategy, but also what their opponent's best strategy. Our example had just one Nash Equilibrium. Some games have many and some have none.

Exercise: Test it out!

Let's see what strategy really is the best.

In groups of three, take a character sheet and two cards and play four rounds of the Prisoner's Dilemma with three other groups. Each member of your group should get to play four rounds while the other two members help record the results.

Each round, place a C (Confess) or D (Deny) card on the table face down. Choose the card based on the instructions on your character sheet. You and your opponent will show your cards at the same time. Your groupmates will record the results below:

		OPPONENT	
		Confess	Deny
YOU	Confess	3, 3	0, 7
	Deny	7, 0	1, 1

For each of your three opponents, write down your opponent's character and the number of detentions you get each round.

Round 1	
Round 2	
Round 3	
Round 4	
TOTAL	

Round 1	
Round 2	
Round 3	
Round 4	
TOTAL	

Round 1	
Round 2	
Round 3	
Round 4	
TOTAL	

Example: Pick a Movie

Manuel and Miranda are going to see a movie together. Manuel wants to watch *Moana* but Miranda wants to watch *Fantastic Beasts and Where to Find Them.* They get separated on their way to the ticket booth and have to each decide which movie ticket to buy. They would each be more satisfied seeing the movie they want but would be unhappy watching a movie alone. The worst situation is if they both go to movie they don't want to see and they are not even together! The payoff table below shows how satisfied (from 0 to 10) each person would be with each situation.

Find what each person's decision should be depending on what the other person chooses. Are there any Nash Equilibriums? If so, circle them.

		MIRANDA Moana	**MIRANDA** Fantastic Beasts
MANUEL	Moana	10, 7	4, 4
MANUEL	Fantastic Beasts	2, 2	7, 10

We see that there are 2 Nash Equilibriums: each person should choose whichever movie the other person picks. The best strategy is to watch a movie together. However, there is no way of knowing what the other person has picked! In this type of game, there is no strategy which ensures that a Nash Equilibrium is reached without knowing the other player's decisions.

However, if Manuel buys his ticket first and Miranda knows what it is, she now has a strategy which ensures that a Nash Equilibrium is reached. Even if they could make the decision together, there is no fair way to decide which movie to see. However, when you play this game with different numbers (different satisfaction levels for different situations), results may be different. For example, if Manuel and Miranda were okay with watching movies separately, the best strategy would be to each see their own preferred movie.

Exercise: Split or Steal

Based on the video we watch in class, fill in the payoff table below:

		IBRAHIM	
		Split	Steal
NICK	Split		
	Steal		

Are there any Nash Equilibirum or a best strategy for either Nick or Ibrahim? What should they do?

Let's watch the rest of the video and see what happens.

What makes this game different from the other simultaneous games we've looked at today?

Sequential Games

In **sequential games**, players make moves one after another. Players know what others have already done in the game. These problems usually cannot be visualized in a straightforward way like simultaneous games and are often more strategically complex.

Example: Counting from 21

With a partner, play the game of Counting from 21 two or three times (switch who goes first each time). Record each turn on the chart on the next play.

Rules:

1. Player A counts down starting from 21. Player A can count 1-4 numbers each turn.
2. Picking up from where Player A left off, Player B continues counting down. Again, saying 1-4 numbers each turn.
3. Players continue alternating turns until one of the players reaches 0. The player who says 0 is the winner!

What do you think is the winning strategy for Player A? What about Player B?

Bonus: Is there a strategy that will guarantee that Player A loses? What about Player B?

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Player B	

Backwards Induction

The key to finding winning strategy in games like Counting from 21 is to think backwards! How do you guarantee that your last move is a winning one? What moves do you have to make before that? Step by step, reason your way from the last move to the beginning of the game to make your first decision.

This is called **Backwards Induction**. Not only will it help you make decisions in games, but it can be applied to solve real world problems.

Example: Dollar Auction

In this game, the bank is auctioning off **$1**. Everyone is allowed to bid as many times as they want with minimum increases of 5 cents per bid.

- The **highest bidder** will get the $1 from the bank in exchange for their bid.
- The **second highest bidder** will also pay their bid to the bank but they get nothing in return.

Let's play! What was the highest bid? What was the second highest bid?

In the strategy for this game, it appears rational at the beginning to bid small amounts since the highest bidder will get a profit as long as their bid is less than $1. However, competition with others will usually drive the bid up above $1. Now, everyone except the bank is losing money!

So why do people keep bidding? Why does this make sense?

This is an example of how a seemingly rational strategy can have an irrational outcome.

Bonus: Make a Game!

1. Grab a partner and together, make up a game with 2 players and 2 decisions each player can make.
2. On a separate sheet of paper, write down the rules to your game at the top of the page.
3. On this handout, fill in the payoff table for your game. Find any Nash Equilibriums and best strategies for each player (if there are any).
4. Switch your rules sheet with another pair of students.
5. Play the new game! See if you can figure out what the best strategy is (if there is one) without making a payoff table.

Problem Set

1. Luna and Hermione are competing in a magic contest by casting spells on a bag of 30 potatoes. Their goal is to keep as many potatoes in their bags as possible. However, they cast their spells at the same time so it interferes with the other witch's potato bag too. The following payoff table tells us the number of potatoes still in each players bag depending on the spells they cast.

		HERMIONE	
		Stupefy	Wingardium Leviosa
LUNA	Stupefy	3, 2	6, 26
	Wingardium Leviosa	10, 11	8, 12

   (a) What should Luna do depending on the spell Hermione might cast?

   (b) What should Hermione do depending on the spell Luna might cast?

   (c) Are there any Nash Equilibriums? What are they?

2. In the game Matching Pennies, Player A and Player B each put a penny on the table at the same time, either showing Heads or Tails. If the two pennies match, then Player A gets to keep Player B's penny. However, if the pennies do not match (one is heads and the other is Tails), then Player B gets to keep Player A's penny.

   (a) Before filling in the payoff table, do you think there will be any Nash Equilibriums?

   (b) Fill in the payoff table on the next page for this game where payoffs represent how many pennies each player gains or loses for each situation. Find each player's best decision depending on what the other player does and mark it on the payoff table.

   (c) Circle any Nash Equilibriums. Explain whether or not there is a best strategy for Player A and/or Player B.

PLAYER B

		Heads	Tails
PLAYER A	Heads		
	Tails		

3. Harry and Ron are playing Chicken on their broomsticks. They fly towards each other on their brooms and can either Swerve before they hit each other or keep going Straight. If they both Swerve, the game is a tie. If one Swerves and the other goes Straight, the one who Swerves is the loser and has to pay the winner 5 galleons. However, if they both go straight, they will crash into each other and break their brooms which cost 100 galleons each!

    (a) Fill in the payoff table for this game. *Hint: Payoffs can be negative.*

RON

		Swerve	Straight
HARRY	Swerve		
	Straight		

    (b) Are there any Nash Equilibriums?

    (c) Is there a best strategy for each player?

4. Zonko's and Weasley's Wizard Wheezes are two competing joke shops. They are each about to release a new product. The payoff table on the next page shows the profit (in galleons per day) the shops would get depending on the products they choose to sell.

    Is there a best strategy or Nash Eqilibrium? (We repeat what we did before for each row and column, considering the outcomes of each possible situation.)

		WEASLEYS' WIZARD WHEEZES		
		Dungbombs	Pygmy Puffs	Fever Fudge
	Dungbombs	3, 3	24, 13	24, 17
ZONKOS	Pygmy Puffs	13, 24	35, 35	50, 17
	Fever Fudge	17, 24	17, 50	48, 48

5. Rock, Paper, Scissors is a game played all the time, often used to choose between different options.

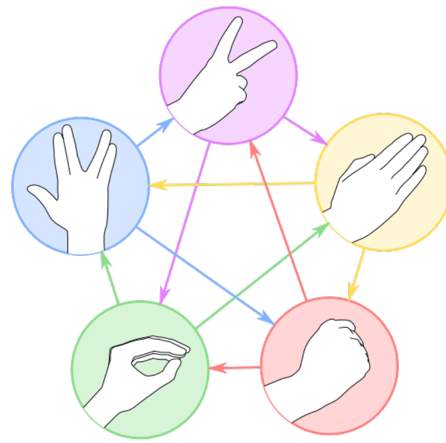
   (a) Is Rock, Paper, Scissors a simultaneous or sequential game?

   (b) Based on the following payoff table, are there any Nash Equilibriums or best strategies for Rock, Paper, Scissors?

		PLAYER B		
		Rock	Paper	Scissors
	Rock	0, 0	-1, 1	1, -1
PLAYER A	Paper	1, -1	0, 0	-1, 1
	Scissors	-1, 1	1, -1	0, 0

   (c) Rock, Paper, Scissors, Lizard, Spock is variation of Rock, Paper, Scissors with 5 different moves you can make. The rules are:

- Scissors cuts paper
- Paper covers rock
- Rock crushes lizard
- Lizard poisons Spock
- Spock smashes scissors
- Scissors decapitates lizard
- Lizard eats paper
- Paper disproves Spock
- Spock vaporizes rock
- Rock crushes scissors



Create the payoff table for Rock, Paper, Scissors, Lizard, Spock.

6. Using the following payoff table, circle any Nash Equilibriums and find the best strategies for Player A and Player B:

PLAYER B

PLAYER A	Up	Down	Right	Left	Front	Back
Up	6, 2	0, 0	3, 5	6, 8	2, 4	4, 1
Down	10, 8	6, 7	6, 3	3, 2	1, 5	9, 6
Right	5, 3	2, 1	7, 6	5, 3	2, 1	6, 5
Left	2, 5	4, 5	8, 7	9, 8	9, 6	5, 1
Front	2, 10	5, 8	5, 3	10, 6	5, 4	5, 3
Back	1, 10	2, 1	2, 3	5, 8	1, 5	8, 5

7. In the game of Tic-tac-toe, two players take turns putting X's and O's on a 3×3 chart. The winner is the first player who is able to make a straight line of three X's or three O's depending on what they are playing as.

   (a) Is this a simultaneous or sequential game?

   (b) Player 1 starts the game by putting an X in the middle spot. What moves could Player 2 make next to give Player 1 a winning strategy? Play this game with a partner a few times to test out your ideas.

8. Take a Square is a sequential game similar to the Counting from 21 game we looked at earlier. Player A starts the game by subtracting a square number from 26. Next, Player B will subtract a square number from what is left after Player A's turn. The 2 players alternate turns and whoever reaches 0 first is the winner. What is the winning strategy for player A?

   Note: A square number is a number equal to a whole number multiplied by itself. For example, $2 \times 2 = 4$ so 4 is a square number. The first 5 square numbers are: 1, 4, 9, 16, and 25.

   Sample Game:

   Player A: $26 - 9 = 17$

   Player B: $17 - 4 = 13$

Player A: $13 - 1 = 12$

Player B: $12 - 9 = 3$

(Now, each player can only subtract by 1 the number will go below 0)

Player A: $3 - 1 = 2$

Player B: $2 - 1 = 1$

Player A: $1 - 1 = 0$ *Player A is the winner!*

9. In the 100 game, 2 players are now alternating turns counting up starting from 0. The goal is to be the player who counts 100. During their turn, each player can count 1-10 numbers.

   (a) Based on the winning strategies for the games Counting from 21 and Take a Square, what would your first instinct be when playing this game as Player A? Write down this first instinct strategy before testing it out and finding the real winning strategy.

   (b) What is the winning strategy for Player A? How does it compare to what you wrote earlier?

   (c) Explain the similarities and differences between the winning strategy for the 100 game, Counting from 21, and Take a Square?

10. **Challenge problem.**
    James and Lily are playing the Magical Centipede game. Here are the rules:

    - The game starts with two candy centipedes.
    - James goes first and gets to choose between the moves Take or Pass. If James Takes, then he and Lily each get one centipede and the game ends. If James Passes, one more candy centipede magically appears and the turn goes to Lily.
    - Now there are three candy centipedes. Lily also gets to choose whether to Take or Pass. If Lily Takes, she gets all the centipedes and the game ends. If Lily Passes, then one more candy centipede magically appears and the turn goes to James.
    - James again gets to choose whether to Take (splitting half the centipedes with Lily) or Pass (increasing the centipedes by one).
    - On her turn, Lily again gets to choose whether to Take (getting all the candy centipedes for herself) or Pass (increasing the centipedes by one).

- The game can go on for 10 turns if James and Lily both Pass at every turn. If at the end of 10 turns, no one has chosen to Take, the pile of now 12 centipedes is split evenly between James and Lily.

James and Lily both love candy centipede so they each want to eat as many as possible!

(a) Write down your first instinct of what James should do.

(b) Using backwards induction, find the strategy that will guarantee James the most magical candy centipedes.