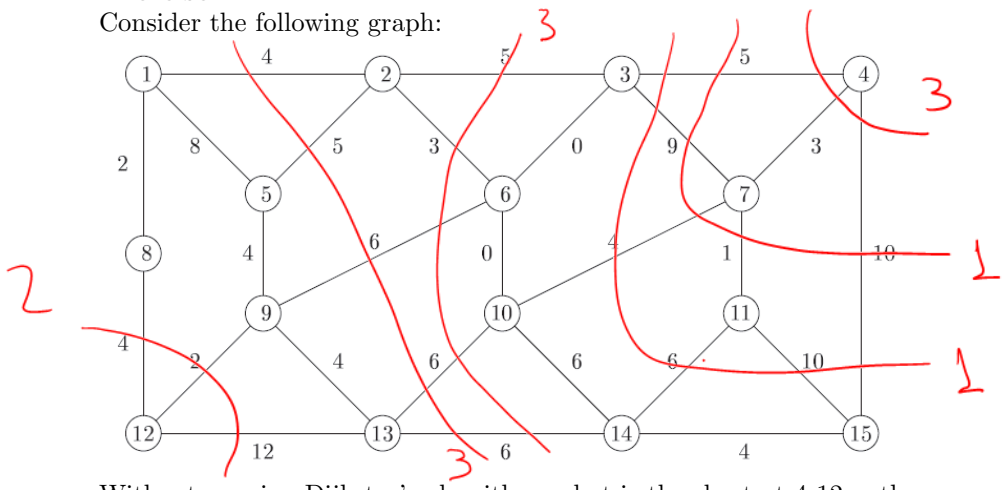
Exercise

Consider the following graph:



Without running Dijkstra's algorithm, what is the shortest 4-12 path you can come up with?

Using barriers, what is the best lower bound you can get on the length of the shortest path?

Solution

Note that the path 4-3-6-9-12 has cost 13

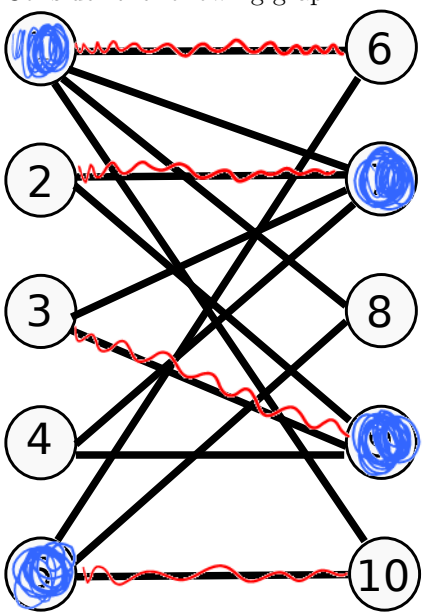
Now consider the following barriers (drawn above):

- $S = \{4\}$, $w(S) = 3$
- $S = \{4, 7\}$, $w(S) = 1$
- $S = \{4, 7, 11\}$, $w(S) = 1$
- $S = \{3, 4, 6, 7, 10, 11, 14, 15\}$, $w(S) = 3$
- $S = \{2, 3, 4, 6, 7, 10, 11, 14, 15\}$, $w(S) = 3$
- $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15\}$, $w(S) = 2$

One can check that the barriers are feasible (that is, the cost of each edge is $\geq$ than the weights of the barriers it crosses). Thus, any path must have length at least 13. Thus the path above is optimal.

Exercise

Consider the following graph:



What is the largest matching you can come up with?

Using vertex covers, what is the best upper bound you can get on the size of the largest matching?

Solution

Consider the match given by $\{1,6\},\{2,7\},\{3,9\},\{5,10\}$. This matching has size 4.

Now consider vertices $1,5,7,9$. It is also easy to check that this is a vertex cover (that is, any edge has an end in at least one of those vertices). Thus any mathing must have size at most 4. So the matching above is optimal.

(we drew the matching in red and vertex cover in blue above)

Exercise

Helen Wheels drives a truck "over the road" between the cities listed in the mileage table below. She would like to visit all the cities listed in the table below. Her route must start and end in Ann Arbor, visiting every city exactly once.

	Ann Arbor	Detroit	Flint	Grand Rapids	Kalamazoo	Lansing
Ann Arbor	*	51	56	146	101	76
Detroit	51	*	62	156	136	90
Flint	56	62	*	121	134	56
Grand Rapids	146	156	121	*	50	91
Kalamazoo	101	136	134	50	*	78
Lansing	76	90	56	91	78	*

What is the shortest route you can come up with?

Using radii, what is the best lower bound you can get on the length of the shortest route?
(in this problem it may not be possible to get the two values to match. Try to get as close as you can)

Solution

Let us say Ann Arbour is city 1, Detroit is city 2, Flint is city 3, Grand Rapids is city 4, Kalamazoo is city 5, Lansing is city 6.

A tour is, for instance, 1,2,3,6,4,5,1, which has cost 411.

(going from 1 to 2 has cost 51, from 2 to 3 has cost 62, from 3 to 6 cost 56, etc)

This is in fact the shortest possible tour, but I did not expect you to get this on your own. The hope was that you made reasonable choices and obtained a tour that is hopefully not to far from this one.

Similarly, a set of radii that works (and in fact is best possible) is:

- $r_1 = 22.5$
- $r_2 = 28.5$
- $r_3 = 33.5$
- $r_4 = 50$
- $r_5 = 0$
- $r_6 = 22.5$

which shows that any tour must have length at least 314.

Again, I did not expect you to get optimal solutions, but to try and get by inspection a reasonably good one.

Note that this shows that we are at most 97 units away from optimal (if we did not know that we already actually have an optimal solution in the first place)