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Senior Math Circles

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Whi so Comple^x Part I

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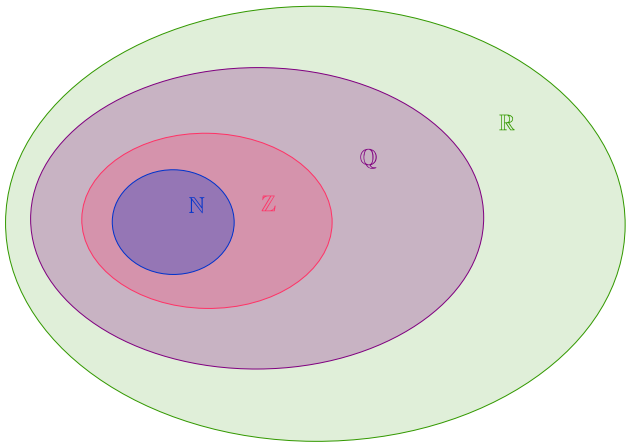
A complex situation

$$\begin{aligned} -1 &= i^2 \\ &= i \times i \\ &= \sqrt{-1}\sqrt{-1} \\ &= \sqrt{-1 \times -1} \\ &= \sqrt{1} \\ &= 1 \end{aligned}$$

Sets as we know them



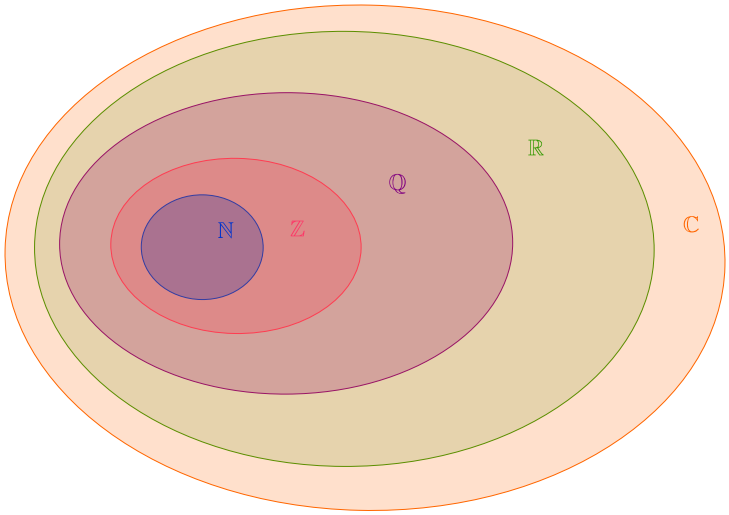
These sets can be thought of as helping us to solve polynomial equations. However, $x^2+1=0$ has no solution in any of these sets.

Defining a complex number

A complex numbers (in standard form) is an expression of the form $x + yi$ where $x, y \in \mathbb{R}$ and i is the imaginary unit. We often use the letter z to express a complex number.
Examples :

- $5 - 6i$
- $\sqrt{3} - i$
- -8
- $\frac{2i}{5}$

A new set!

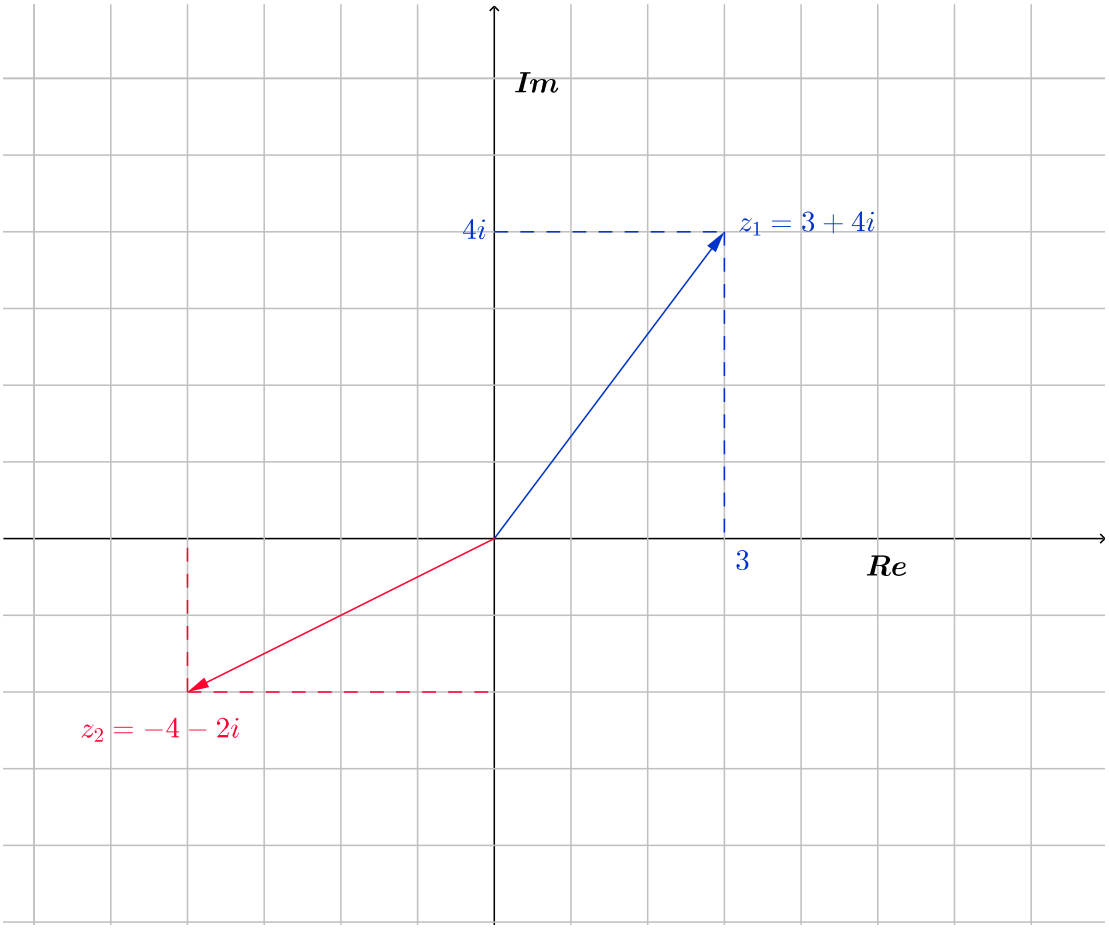


How to represent a complex number?

We know that every point on the number line represents a real number. But how can we represent $3 + 4i$?

Solution: we plot $z = x + yi$ as a point (x, y) in a cartesian plane, called complex plane, where the $x - axis$ is the real axis and the $y - axis$ is the imaginary axis. Such a diagram is called an Argand Diagram.

The Argand Diagram



The Modulus

The **modulus (or absolute value)** of a complex number $z = x + yi$ is the nonnegative real number :

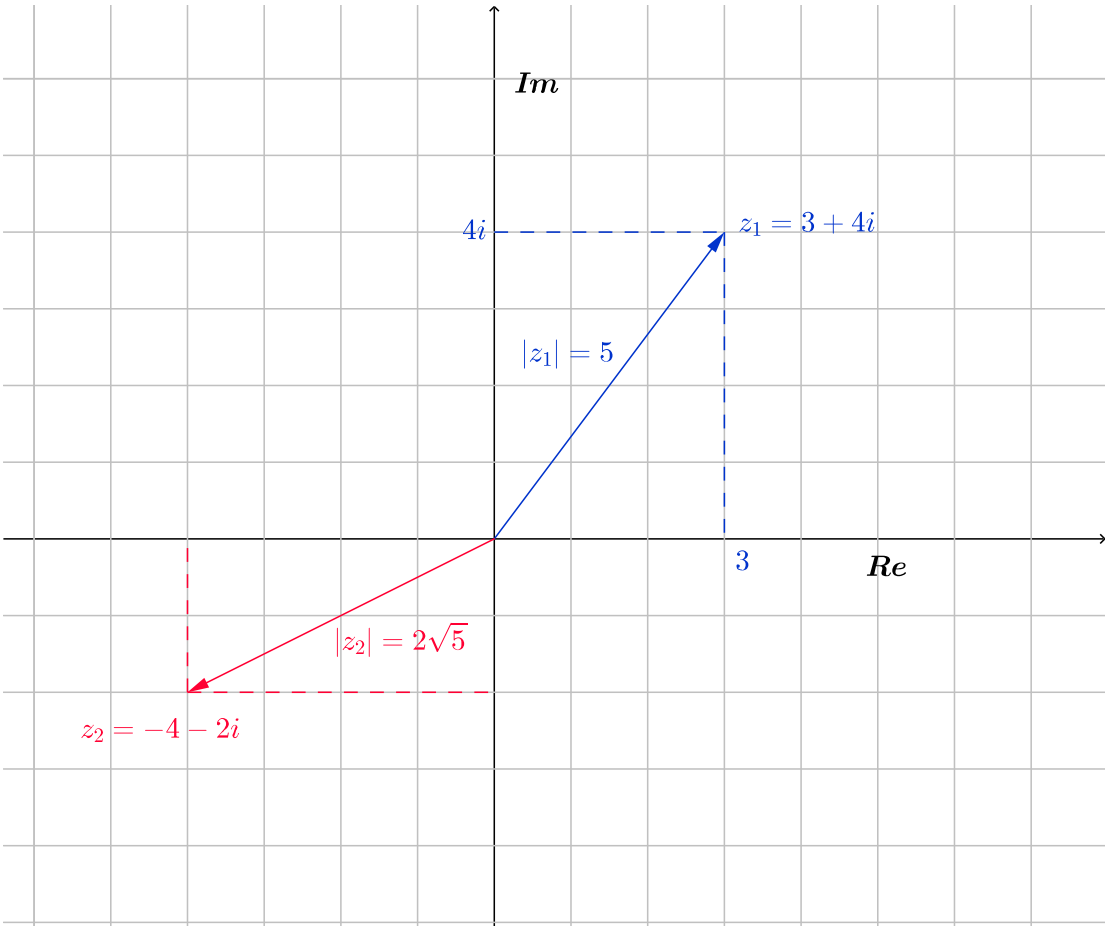
$$|z| = |x + yi| = \sqrt{x^2 + y^2}$$

Compute the modulus of the following complex numbers:

1. $z_1 = 3 + 4i$
   Solution: $|z_1| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$

2. $z_2 = -4 - 2i$
   Solution: $|z_2| = \sqrt{(-4)^2 + (-2)^2} = \sqrt{20} = 2\sqrt{5}$

What is the geometrical interpretation of the modulus?



The Complex Conjugate

The **complex conjugate** of a complex number $z = x + yi$ is the complex number :

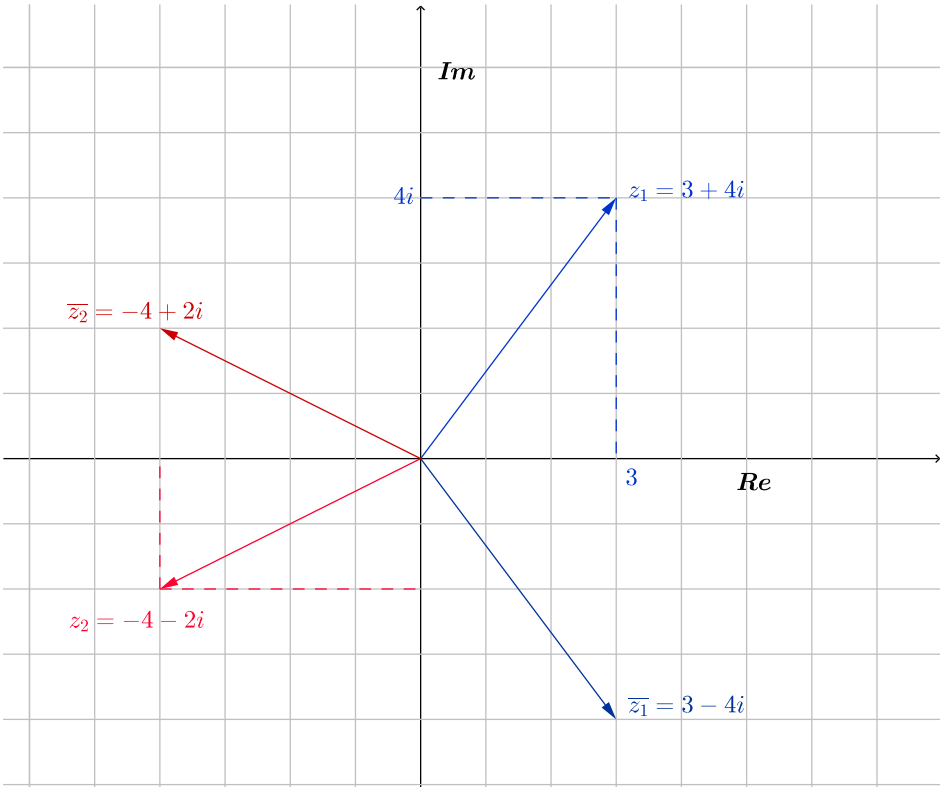
$$\overline{z} = x - yi$$

Detemine the conjugate of each of those complex numbers:

1. $z_1 = 3 + 4i$
   Solution: $\overline{z_1} = 3 - 4i$

2. $z_2 = -4 - 2i$
   Solution: $\overline{z_2} = -4 + 2i$

What is the geometrical interpretation of the complex conjugate?



Complex Arithmetic

Let $z = a + bi$ and $w = c + di$. We define :

- $z + w = (a + bi) + (c + di) = (a + c) + (b + d)i$
- $z - w = (a + bi) - (c + di) = (a - c) + (b - d)i$
- $z \cdot w = (a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i$

Using the definition of the multiplication, show that $i^2 = -1$

$$i^2 = (0 + i)(0 + i) = (0 \cdot 0 - 1 \cdot 1) + (0 \cdot 1 + 1 \cdot 0) = -1 + 0i = -1$$

Hence i is one complex square root of -1. The other square root of -1 is $-i$.

Examples : Let $z_1 = 3 + 4i$ and $z_2 = -4 - 2i$, express each of the following in standard form:

1. $z_1 + z_2$
   $(3 + 4i) + (-4 - 2i) = (3 - 4) + (4 - 2)i = -1 + 2i$

2. $z_1 - z_2$
   $(3 + 4i) - (-4 - 2i) = (3 + 4) + (4 + 2)i = 7 + 6i$

3. $z_1 z_2$
   $(3+4i)(-4-2i) = (3(-4)-4(-2))+(3(-2)+4(-4))i = (-12+8)(-6-16)i = -4-22i$

4. $z_1 z_2^2$
   $(z_1 z_2)z_2 = (-4 - 22i)(-4 - 2i) = (16 - 44) + (8 + 88)i = -28 + 96i$

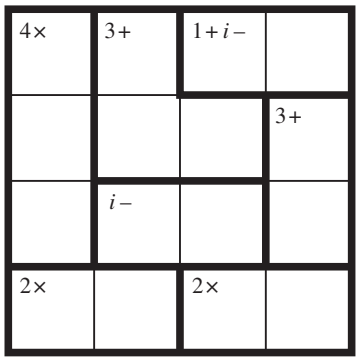
Consider :

$$\frac{3 + 4i}{-4 - 2i}$$

What does it mean ?

$$\frac{3 + 4i}{-4 - 2i} \cdot \frac{-4 + 2i}{-4 + 2i} = \frac{-20 - 10i}{20} = \frac{-2 - i}{2}$$

Complex KenKen

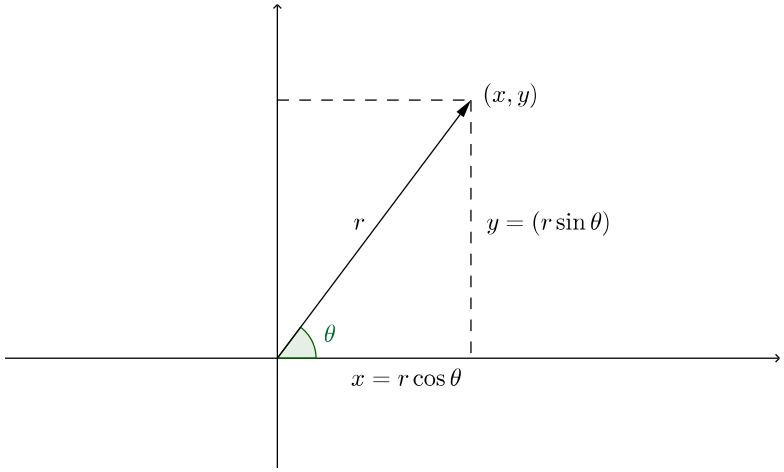


Solve using $1, 2, 1 + i, 1 - i$

Source: The College Mathematics Journal, Volume 43, Number 1, January 2012, pp. 37-42(6)

Polar Coordinates

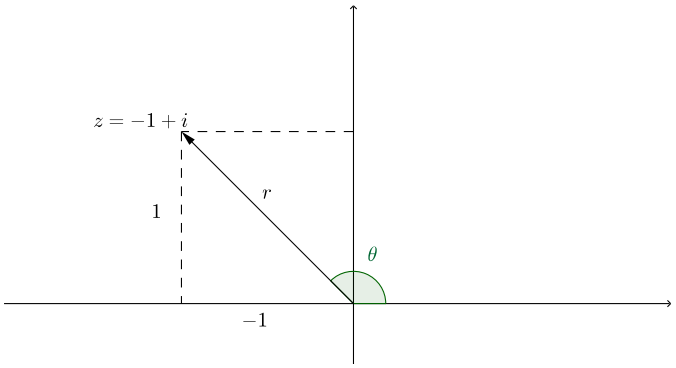
A point in the plane corresponds to a length and an angle:



Example : Let $P(3, \frac{\pi}{4})$. What are the cartesian, (x, y) coordinates of P?

$$P(x, y) = P\left(3\cos\frac{\pi}{4}, 3\sin\frac{\pi}{4}\right) = P\left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$$

Polar Coordinates in the Complex Plane



What would be the polar coordinate $P(r, \theta)$ for the complex number $z = -1 + i$?

$$r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$

$$\theta = \arctan\left(\frac{1}{-1}\right) = \arctan(-1) = \frac{3\pi}{4}$$

Complex Numbers in Polar Form

This polar representation gives another way of expressing a complex number.
Let $z = x + yi$, then:

$$\begin{aligned} z &= x + yi \\ &= r\cos\theta + r\sin\theta i \\ &= r(\cos\theta + i\sin\theta) \end{aligned}$$

$z = r(\cos\theta + i\sin\theta), r > 0$ is called the **polar form** of z, where r is the modulus of z and the angle θ is called the argument of z. Express each complex number in polar form:

1. $z = i$
   $r = \sqrt{1^2 + 0^2}, \theta = \frac{\pi}{2}$
   $z = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

2. $w = -4$
   $r = 16, \theta = \pi$
   $w = 16(\cos\pi + i\sin\pi)$

3. $v = \sqrt{3} - 3i$
   $r = \sqrt{3+9} = \sqrt{12} = 2\sqrt{3}, \theta = \frac{5\pi}{3}$
   $v = 2\sqrt{3}\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$

Comparing Complex Numbers

For two complex numbers $z = x + yi = r_1(\cos\theta + i\sin\theta)$ and $w = a + bi = r_2(\cos\phi + i\sin\phi)$, $z = w$ only if :

- $x = a$ and $y = b$ or
- $r_1 = r_2$ and $\theta = \phi + 2\pi k, k \in \mathbb{Z}$

Example : For which value of a and b is the following true:

$$\begin{aligned} (a+b)(5-i) &= 10a + b + 1 - (2a+1)i \\ 5a + 5b - ai - bi &= 10a + b + 1 - (2a+1)i \\ 5a + 5b - (a+b)i &= 10a + b + 1 - (2a+1)i \end{aligned}$$

Multiplication of Complex Numbers in Polar Form

Let $z = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ and $w = 3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$.

What is the polar form of zw?

Consider $v = r_1(\cos\theta + i\sin\theta)$ and $t = r_2(\cos\gamma + i\sin\gamma)$, then :

$$\begin{aligned} vt &= r_1(\cos\theta + i\sin\theta)\cdot r_2(\cos\gamma + i\sin\gamma) \\ &= r_1r_2(\cos\theta + i\sin\theta)(\cos\gamma + i\sin\gamma) \\ &= r_1r_2(\cos\theta\cos\gamma + i\sin\theta\cos\gamma + i\cos\theta\sin\gamma + i^2\sin\theta\sin\gamma) \\ &= r_1r_2(\cos\theta\cos\gamma - \sin\theta\sin\gamma + i\sin\theta\cos\gamma + i\cos\theta\sin\gamma) \\ &= r_1r_2(\cos\theta\cos\gamma - \sin\theta\sin\gamma + i(\sin\theta\cos\gamma + \cos\theta\sin\gamma)) \end{aligned}$$

Using the trigonometric identities for $\cos(A+B)$ and $\sin(A+B)$, we get :

$$vt = r_1r_2(\cos(\theta+\gamma) + i\sin(\theta+\gamma))$$

In short, to multiply two complex numbers in polar forms, we multiply the modulus and we add the arguments! Let $z = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ and $w = 3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$. What is the polar form of zw?

$$zw = 6\left(\cos\left(\frac{2\pi}{3} + \frac{\pi}{6}\right) + i\sin\left(\frac{2\pi}{3} + \frac{\pi}{6}\right)\right)$$

$$zw = 6\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$$

If $z = r(\cos\theta + i\sin\theta)$ is a complex number, geometrically, where would iz be in relations to z ? $i = 0 + 1i$ in polar form :

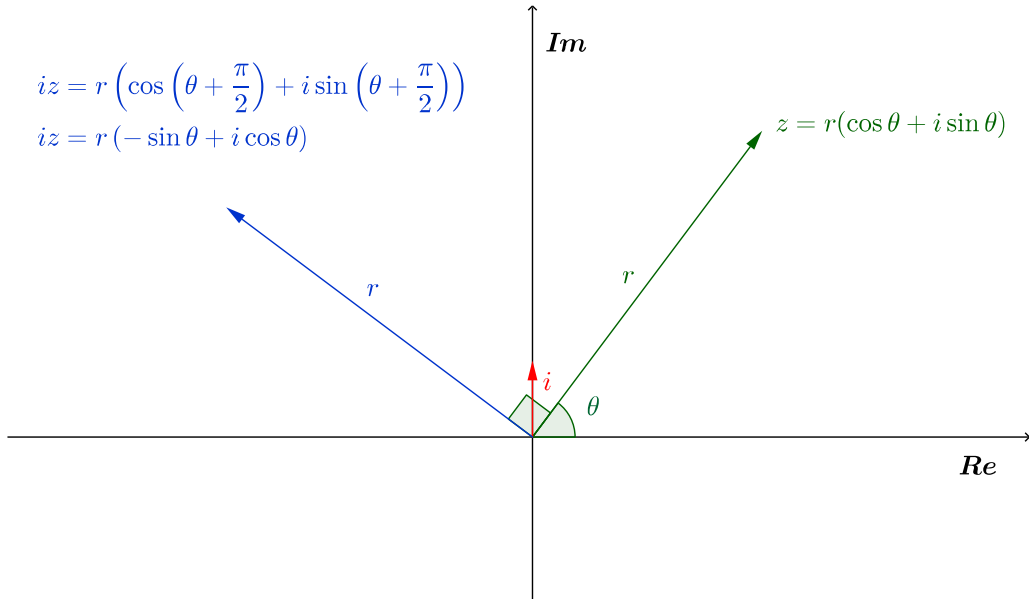
- $r = 1$
- $\theta = \dfrac{\pi}{2}$

$$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

So :

$$iz = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\cdot r(\cos\theta + i\sin\theta) = r\left(\cos\left(\theta + \frac{\pi}{2}\right) + i\sin\left(\theta + \frac{\pi}{2}\right)\right)$$

It's a 90 degrees rotation!



De Moivre's Theorem

If $z = r(\cos\theta + i\sin\theta)$, then $z^n = r^n(\cos n\theta + i\sin n\theta)$.

Example : What is the value of $\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)^{1000}$

$$\begin{aligned}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)^{1000} &= \left(\cos\frac{3000\pi}{4} + i\sin\frac{3000\pi}{4}\right)\\ &= (\cos 750\pi + i\sin 750\pi)\\ &= \cos 0 + i\sin 0\\ &= 1\end{aligned}$$