Math Circles - Surfaces

Night 3

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17th February 2016

Last time we talked about words, cutting and pasting, orientability, and established the following result.

Theorem 1. *If A and B are any two surfaces, then $\chi(A\#B) = \chi(A) + \chi(B) - 2$.*

Pretty neat hey? It finally feels like we're starting to make some progress.

What we have so far

Phew! Okay, let's stop and take a look at what we know so far. Using our formula for the connected sum and the Euler characteristic, as well as our newfound ability to spot whether or not something is orientable just by looking at a word representation or a planar model, we can make the following table. It's a great exercise to verify all the information below.

Surface	Euler Characteristic	Orientable?
Sphere	2	Y
Torus	0	Y
2-holed torus ($T\#T$)	-2	Y
n-holed torus	2 - 2n	Y
Klein Bottle	0	N
Projective Plane	1	N
$PP\#PP$	0	N
$PP\#PP\#PP$	-1	N
$PP\#KB$	-1	N
$PP\#T$	-1	N

Looking at this table we can now conclude a lot of things! The torus is indeed non equivalent to the Klein bottle (phew!), and the n-holed torus is not equivalent to the k-holed torus for any $k \neq n$. However, we still can't tell apart $PP\#PP$ and KB or $PP\#PP\#PP$, $PP\#T$ and $PP\#KB$. Maybe they're the same, or maybe we need more invariants. We will soon find out.

Cutting and Pasting with Words

Earlier on, we showed that the surface corresponding to the word $abca^{-1}b^{-1}c^{-1}$ is similar to the torus through a series of cutting and pasting polygons. Although this method in theory is useful, in practice it is cumbersome. We'd like to encode some of this in the form of legal word manipulations. Before we try to list what exactly we can do with words that keeps the corresponding surface the same, we should note a few more things. It doesn't matter where on the planar model we start our word, or whether we go clockwise or counter-clockwise. These kinds of things should be accounted for in a useful list of legal word manipulations. Let's try and compile such a *list of legal word manipulations.*

1. **Relabel edges:** For any particular edge, you can name it whatever you want. You can relabel an edge named a, "Bernice" if you want, just as long as you relabel both occurrences of the original a.

2. **Change direction:** For any particular edge, you can switch the direction of both occurrences of the same letter. For example, $aabbcc \sim aabbc^{-1}c^{-1}$.

3. **Cycle word:** As we noted, it doesn't matter where on a planar model you start reading off the labels of the edges to form your word. So, we're allowed to cycle words and, for example, we have that $abca^{-1}c^{-1}b^{-1} \sim c^{-1}b^{-1}abca^{-1}$.

4. **Read in the opposite direction:** This corresponds to reading around a planar model counter-clockwise instead of clockwise (say).

5. **Merge edges:** If the same sequence of letters appears as a group, then you can merge them into one. For example, $\ldots bc \ldots bc \ldots \sim \ldots d \ldots d \ldots$.

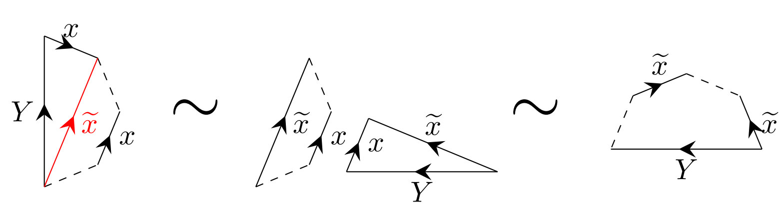
6. **Cancel aa^{-1} pairs (unless that's all you have):** This is what you'd expect and is what we did in the last step of showing $abca^{-1}b^{-1}c^{-1} \sim T$ above.

7. $\ldots Yxx \ldots \sim \ldots xY^{-1}x \ldots$.

8. $\ldots Yx \ldots x^{-1} \ldots \sim \ldots x \ldots x^{-1}Y \ldots$.

In 7 and 8, the capital Y corresponds to any sequence of letters. The legal manipulations 1 to 6 are all manipulations that are clear are legal. However, 7 and 8 do require some non-trivial cutting and pasting to guarantee that we don't start with a word, apply (say) 7 and end up with a word corresponding to a surface not equivalent to the original surface. In fact, making sure our words don't change enough to change the surface is the whole point of these rules. Let's try proving 8.

Proof. Let's say we have a planar model which is a polygon that has a word representative looking like $\ldots Yx \ldots x^{-1} \ldots$ where Y represents a whole bunch of edges. Then we can cut and paste that polygon as follows where the dashed lines represent other edges in the polygon which we're not particularly interested in, and we are cutting along the red edge.



The proof of rule 8

We can now see that the resulting polygon has a word representation of the form $\ldots x \ldots x^{-1}Y \ldots$, which completes the proof. ■

Rule 7 is proved in pretty much the same way, you just have to make a clever cut and paste. Try it for yourself!

The Manipulations in Action

So we've now created these word manipulations, which we've checked correspond to cutting and pasting planar models (which really was far too cumbersome for my liking). What these word manipulation rules have allowed us to do, is to perform the usual cutting and pasting and encode it in words! Hopefully we will now find it much easier to check whether two surfaces are the same now. Let's put it into action and see!

Let's try and attack the question of whether or not $KB \sim PP\#PP$. We have

$$KB \sim aba^{-1}b \sim aabb \sim PP\#PP$$

where the manipulation used rule 7, and we see that $KB \sim PP\#PP$! Amazing hey? See if you can perform the cut and paste (we only did it once, so there's only one) that turns the Klein bottle into the connected sum of two projective planes. This is promising, let's see what else we can work out! Maybe we can settle this question about whether or not $PP\#KB \sim PP\#T \sim PP\#PP\#PP$.

$$\begin{aligned} PP\#T \sim aabcb^{-1}c^{-1} &\overset{7}{\sim} ab^{-1}acb^{-1}c^{-1} \\ &\overset{7}{\sim} ab^{-1}b^{-1}c^{-1}a^{-1}c^{-1} \\ &\overset{7,3}{\sim} aab^{-1}b^{-1}c^{-1}c^{-1} \\ &\overset{1}{\sim} aabbcc \sim PP\#PP\#PP. \end{aligned}$$

Whoa, now there's something unexpected! Let's stop and think about this for a sec. Knowing that $T\#PP \sim PP\#PP\#PP$ should freak you out a little (a lot really). What it tells us is that if we take a nice orientable surface like a torus, and then attach a projective plane, the non-orientability of the projective plane spreads through the torus, in some sense turning it into two projective planes! Weird. Another thing this tells us is that connected sum is not cancellative, that is, just because $A\#C \sim A\#B$ it **does not** mean $B \sim C$ (since we know $PP\#PP \not\sim T$).

Now we know from above that $KB \sim PP\#PP$, so we would expect that $KB\#PP \sim PP\#PP\#PP$, but let's check anyway.

$$PP\#KB \sim aabcbc^{-1} \overset{7}{\sim} aabbc^{-1}c^{-1} \sim PP\#PP\#PP.$$

Through these manipulations, we now know $PP\#PP \sim KB$ and, more surprisingly, $PP\#PP\#PP \sim KB\#PP \sim T\#PP$. Note here that both $PP\#PP$ and KB have the same Euler characteristic and orientability, as do $PP\#PP\#PP$, $T\#PP$, and $KB\#PP$. Coincidence? Maybe.

Big Questions - One More Time

So, we've been stumbling around in the dark for a while now, and it feels like we've found a few light switches. Our shins are getting sore from banging into all the furniture in the dark, so let's regroup and revisit the big questions we had at the beginning of our journey.

1. If I give you any two surfaces, can you tell me whether or not they are equivalent?
2. If I give you a surface, can you tell me which one it is?
3. Can we make a list of all possible surfaces?

Looking back at all the work we've done, cutting and pasting, developing the Euler characteristic and orientability, it seems like if the answer to these questions is still "no", we've at least made some progress. Let's try and investigate it further. Suppose we are given a random word corresponding to some surface. Absolutely any word at all, say $abcd^{-1}b^{-1}zz^{-1}wxwx^{-1}$.... Well, let's try and find some sort of standard form for it. We can do this in the following steps:

1. Collect like terms.
Say we have a word $WaXaY$ where W, X and Y are arbitrary strings of letters corresponding to multiple edges. Then by applying rule 7 repeatedly we have

$$WaXaY \sim WaaX^{-1}Y \sim aW^{-1}aX^{-1}Y \sim aaWX^{-1}Y$$

which tells us that if we have like terms, we have that it is equivalent to $PP\#\{\text{something}\}$.

2. Repeat 1 until you can't.
We can look for more like terms in the {something} and repeat the process above. Once we have run out of like terms, our word looks like

$$a_1a_1a_2a_2 \cdots a_na_nB$$

where B is some word without like terms, so it corresponds to an orientable surface. So we have that our surface is equivalent to $\overbrace{PP\#PP\# \cdots \#PP}^{n\text{-times}}\#\{\text{something orientable}\}$. If our word didn't have any like terms to begin with, we just skip the first two steps and go straight to step 3.

3. Group linkings of pairs.
So now we have gotten rid of all the edges which have the same sign appearing in both occurrences of their letter, we can start looking for pairs of edges which are interlocking. We are after things of the form $VaWbXa^{-1}Yb^{-1}Z$, where once again V, W, X, Y, Z are arbitrary strings of letters, which may or may not correspond to multiple edges. Then, using 8 repeatedly and sometimes cycling the word we get

$$\begin{aligned} VaWbXa^{-1}Yb^{-1}Z &\sim aWbXa^{-1}VYb^{-1}Z \\ &\sim abXa^{-1}VYb^{-1}WZ \\ &\sim aba^{-1}VYXb^{-1}WZ \\ &\sim aba^{-1}b^{-1}WZVYX \end{aligned}$$

which tells us that our original surface is

$$\overbrace{PP\#PP\# \cdots \#PP}^{n\text{-times}}\#T\#\{\text{some other orientable word}\}.$$

4. Repeat 3 until you can't.
Enough said. If we keep performing the steps in 3, we end up with a word representing a surface which is equivalent to

$$PP\# \cdots \#PP\#T\# \cdots \#T\#\{\text{some orientable word without pairs of interlocked edges}\}.$$

With a bit of thought you can convince yourself that if we have an orientable word where there are no interlocking edges, then it must be of the form $xx^{-1}yy^{-1}zz^{-1}$... for all the left over edges.

5. Cancel all remaining pairs.
So now we have a word which represents $PP\#\cdots\#PP\#T\#\cdots\#T$, which is pretty amazing. This tells us that whatever surface we have, if it is not a sphere (in which case we would be skipping right to step 5) then it is a connected sum of projective planes and tori! This is flabbergasting, but we can do even better.

6. Let the projective planes work their magic.
We know from earlier that $PP\#T \sim PP\#PP\#PP$, so every time we see $PP\#T$, we can replace it with $PP\#PP\#PP$.

So, if a surface is orientable, then we skip steps 1 and 2, and stop at step 5 (since step 6 doesn't have a chance to go into action) so we end up with a connected sum of tori or a sphere (the latter occurs if there are no linkings of pairs to begin with in step 3). If a surface isn't orientable, then the existence of one projective plane will spread like a virus and the surface will be equivalent to a connected sum of projective planes. Putting all of this together we get the following absolutely remarkable theorem, which was proved sometime around 1920.

Theorem 2 (Classification Theorem of Closed 2-Manifolds)**.** *Any 2-dimensional surface is equivalent to exactly one of the following three surfaces.*

- *A Sphere.*
- *A connected sum of projective planes (if it is non-orientable).*
- *A connected sum of tori (if it is orientable).*

With a bit of careful thought we can now convince ourselves that knowing the Euler characteristic and orientability of a surface is enough to tell us exactly what it is equivalent to!

Victory Lap

From stumbling around in the dark at the beginning, we have come a long way and now our entire universe is illuminated, or at least the parts we were interested in. With the classification theorem in our back pockets, we have the ultimate flashlight to enlighten us about any surface we are given.

This theorem to me is absolutely remarkable, and one of the most beautiful in all of mathematics. The fact that there is such a clean and neat classification of every possible surface blows my mind every time I think about it. It is results like this that makes math worth doing for its own sake.

Although we have been shown a clear view of what's going on, it is by no means the end of the road for studying surfaces. We can always ask more questions (as with anything in mathematics), like what 3-dimensional surfaces can exist? What about 4-dimensional, or even higher? How would we even define such a thing, and can we have some sort of "planar model" of these to help us?

If you are interested in learning more about this area of mathematics (topology and geometry), then a good place to start is the book *The Shape of Space* by Jeffrey Weeks. Good luck!