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Grade 6 Math Circles

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Mechanics & Dynamics

This lesson builds on the Kinematics lesson from February 16/17 2016. It is strongly recommended that you read through that lesson first if you are unfamiliar with kinematics. **Dynamics** is the study of forces that act on physical objects to result in motion. Together, kinematics and dynamics make up the fundamental basis of physics called **mechanics**.

Solving Equations

In this section, we're going to learn to rearrange equations to solve for an unknown. These equations will involve four types of **operations**: addition, subtraction, multiplication, and division.

Exercise: fill in the box to make the equality true.

(a) $5 + \square = 10$

(b) $3 - \square = 2$

(c) $\square + 2 = 9$

(d) $\square \times 3 = 18$

(e) $18/\square = 2$

Instead of drawing the square everytime, we use **variables** which do the same thing. Variables are represented by letters in the alphabet and, just like the squares in the questions above, they indicate an open space where we can place any number that will make the equality true.

Example 1:

In the equation $5 \times m = 10$, we say that $m = 2$ because replacing m with 2 in the equation "satisfies the inequality":

$$5 \times m = 10$$
$$5 \times 2 = 10$$
$$10 = 10 \text{ which is true!}$$

We can use *any* letter we like for the variable.

Exercise: Determine the value of each variable. The first one is done for you.

(a) $5 \times m = 10 \implies m = 2$

(b) $12 + A = 15 \implies$

(c) $3 - 1 = x \implies$

The last one was a lot easier than the other ones. In fact, it's clear that solving for a variable is easier when the variable is on its own on one side of the equation because all you have to do to find its value is perform the correct operations (in this case, $3 - 1$) on the opposite side of the equality.

This will be our general technique for solving equations like this, especially as they get more complicated: we will try to get the variable on its own side of the equation.

Example 2: Solve $x + 2 = 5$.

To get x on its own, we have to subtract 2 from the left hand side (LHS) of the equation because $x + 2 - 2 = x$. However: whatever we do to one side of the equation, we have to do to the other side as well so that the equality will continue to hold. Since we subtracted 2 from the LHS, we have to subtract 2 from the right hand side (RHS):

$$\begin{aligned} x + 2 &= 5 \\ x + 2 - 2 &= 5 - 2 \\ x &= 5 - 2 \\ x &= 3 \end{aligned}$$

Does this hold? If I replace x with 3 in my original equation, $x + 2 = 5$ becomes $3 + 2 = 5$ or $5 = 5$, which is true! So $x = 3$.

Aside: $5 = 5$ is true. If I subtract 1 from the left side: $5 - 1 = 5$ becomes $4 = 5$ which is no longer true unless I also subtract 1 from the right side: $4 = 5 - 1$ becomes $4 = 4$, which is true again!

Example 3: Solve $x - 2 = 5$

To get x on its own on the LHS, we need to add 2: $x - 2 + 2 = x$. So we also have to add 2 to the RHS:

$$\begin{aligned} x - 2 &= 5 \\ x - 2 + 2 &= 5 + 2 \\ x &= 5 + 2 \\ x &= 7 \end{aligned}$$

We can also solve equations which involve multiplication and division:

Example 4: Let $5n$ mean $5 \times n$. Solve $5n = 15$.

To get x on its own, I have to divide the LHS by 5 since: $5n/5 = n$. So I also have to divide the RHS by 5:

$$\begin{aligned} 5n &= 15 \\ \frac{5n}{5} &= \frac{15}{5} \\ n &= \frac{15}{5} \\ n &= 3 \end{aligned}$$

So $n = 3$. Does this work? If I replace n with 3 in our original equation, I get that $5n = 15$ becomes $5 \times 3 = 15$ which then becomes $15 = 15$, which is true!

Example 5: Solve $x/2 = 4$

I have to multiply the LHS by 2 to get x on its own because $\frac{x}{2} \times 2 = x$. That means I also have to multiply the RHS by 2:

$$\begin{aligned} \frac{x}{2} &= 4 \\ \frac{x}{2} \times 2 &= 4 \times 2 \\ x &= 4 \times 2 \\ x &= 8 \end{aligned}$$

Example 6: Solve $5n + 2 = 22$.

Whenever we have an equation involving 2 or more operations, we do any addition or subtraction operations first and any multiplication or division operations last. In this example, I subtract 2 from both sides *before* I divide both sides by 5:

$$\begin{aligned} 5n + 2 &= 22 \\ 5n + 2 - 2 &= 22 - 2 \\ 5n &= 20 \\ \frac{5n}{5} &= \frac{20}{5} \\ n &= \frac{20}{5} \\ n &= 4 \end{aligned}$$

Check if this is correct by replacing n with 4 in our original equation.

Exercise: Solve for x. Remember, $5n = 5 \times n$.

(a) $x + 6 = 12$

(b) $x - 4 = 8$

(c) $4 - x = 2$

(d) $3x = 18$

(e) $4x = 16$

(f) $x/3 = 4$

(g) $x/5 = 5$

(h) $3x + 2 = 20$

(i) $3x - 2 = 10$

(j) $6x + 5 = 23$

(k) $x/2 - 4 = 16$

(l) $x/2 + 7 = 11$

(m) $x/3 - 3 = 3$

$d = vt$: Unknowns in the Real World

In the velocity section of the Kinematics lesson, we introduced the equation $v = \frac{d}{t}$. If we were finding the average speed of an object, we would say v was the average speed, d was the total distance traveled, and t was the time of motion. If we instead wanted the average velocity, we said that v was the average velocity, d was the displacement of the object, and t again was the total time of motion.

Multiplying both sides by t, we get a cleaner looking equation: $d = vt$. We can solve for distance, displacement, average velocity, average speed, *or* time with this equation! **Remember, d is displacement as long as v is average velocity and d is total distance as long as v is average speed**. t is always time.

As long as we are given two of the values d, t, or v, we can solve for the third value.

Example 1:

My friend Julian leaves his house at 2pm to go to the beach 90km away. He wants to get there before low tide at 3:15pm, when all the good surfin' waves are gone. How fast should he drive? Give your answer in km/h.

Begin these questions by writing down what information is provided to you by the problem. We are given a distance here: 90 km (no direction). We are also given a time, which we want in hours: from 2pm to 3:15pm. This is 1 hour and 15 minutes. 15 minutes is 0.25 hours (one quarter of an hour), so the total time that Julian has to get to the beach is 1.25 hours. Since 90 km is a distance, we can find the *average speed* using the relationship $d = vt$. First, we replace the variables with the information we have:

$$d = vt$$
$$90 \text{ km} = v \times 1.25 \text{ hrs}$$

We divide each side by 1.25 hrs to get the average speed on its own:

$$v = \frac{90 \text{ km}}{1.25 \text{ hrs}}$$
$$v = 72 \text{ km/hr}$$

So Julian has to drive 72 km/h to get to the beach on time.

We always include the units when making these calculations. The reason is, we want to make sure that we are not adding quantities with different units (for example: What is 90 seconds + 3 km? Nothing! What is 5 m + 10 km? You have to convert to the same units before you can add this). Notice we had to do this when we converted 1 hour and 15 minutes to 1.25 hours.

What is dynamics?

So far, we've analyzed the movement of objects without considering what causes those movements. When a pencil rolls off a table without being touched, what caused it to move? Why does it tend to *fall* without provocation? Why doesn't it fly up instead? How come my stapler sits on top of the table and doesn't fall through? In my opinion, questions like that are much more interesting than the question of how far the pencil fell or how fast it was moving. To figure out what made the pencil move, we will now delve into the study of dynamics, which focuses on the affects of force on the motion of physical objects.

Mass & Force

We use the concept of **mass** to describe how much matter is inside a physical object. Anything that you can touch has matter in it. Generally, the bigger an object is, the more mass it has, but not always. If you could blow up a balloon to be the same size as an elephant, the balloon would still have less matter (or "stuff") inside it than the elephant does. Although the balloon and the elephant are the same size, we say that the elephant has more mass.

We often weigh objects to figure out what their mass is. In the above example, the elephant would weigh more than the balloon, and that would be one indication that the elephant has more mass than the balloon. Having weight is a *result* of having mass. Weight and mass are not the same thing, though. Mass is a scalar quantity measured in kilograms (kg) and weight is a consequence of gravity acting on mass (see Mechanics Activity #4).

A **force** is a push or a pull on an object. They can be balanced or unbalanced by other forces. Take this example: if you push on a box, then you are applying a force onto that box that is causing it to move. We say that the force of your push is unbalanced. If you and your friend pushed on the box from opposite directions and each of you is pushing just as hard as the other, the box would not move. We say that the force of your push is balanced out by the force of her push. If you pushed harder than your friend, then the box would move again in the direction that you were pushing. We say that the force of your push is unbalanced once again. **Gravity** is one example of a force. It is pulling you down toward the Earth.

Force is a vector quantity measured in Newtons (N), named for Sir Isaac Newton.

Newton's Laws of Motion

Sir Isaac Newton was an English physicist and mathematician who is often credited with "discovering" gravity, but he did much more than that. In fact, Newton almost single-handedly invented the field of physics as we know it today. His three laws of motion are at the core of classical physics. In physics, a **law** is an undisputed truth that describes how the universe behaves under certain conditions.

Newton's First Law

"An object at rest stays at rest and an object in motion stays in motion with the same speed and in the same direction unless acted upon by an unbalanced force."

In simpler terms: if an object is sitting still, you must push it or pull it to make it move. If somebody starts pushing or pulling in the opposite direction as you with the same amount of strength, then the object will once again stop moving.

If instead the object is moving, it will be doing so with constant speed and in the same direction throughout its movement. If you want to do any of the following things to it, you must push it or pull it:

1. Make it go faster
2. Make it go slower
3. Make it change direction

By this logic, if you observe that an object is speeding up, slowing down, or changing direction, we say that an *unbalanced* force is acting on it and causing it to **accelerate**.

Any object with zero acceleration will be moving with constant speed and direction (or not moving at all).

This law is also known as the Law of **Inertia**. Only objects with mass (stuff you can touch) experience inertia.

Newton's Second Law

"F=ma"

Above, we saw that force is related to acceleration: you must apply an unbalanced force to an object with mass in order to make it move or to affect its movement in any way. This relationship is summarized in the equation above. F is the unbalanced force on the object with mass m. This mass will have acceleration a. If you are given any two of the three quantities force, mass, or acceleration, you can solve for the third quantity using this relationship. In this equation, force is measured in N, acceleration in m/s^2, and mass in kg. Recall the equation $d = vt$. Newton's Second Law is solved in the same way.

Example 1:

You and your friend are still pushing that box from before. If you push to the right with a force of 40 N and they push to the left with a force of 20 N, what is the mass of the box? Assume that the box is accelerating at 2 m/s^2 [right].

The unbalanced force is going to be the bigger force (your force) minus the smaller force (your friend's force). Because your force is bigger, the unbalanced force will also be in the same direction as your force: right. We can say that the unbalanced force is therefore 40 N – 20 N = 20 N [right]. We plug this into Newton's second law to get:

$$\begin{aligned} F &= ma \\ 20\text{ N [right]} &= m \times 2\text{ m/s}^2\text{ [right]} \\ 20/2 &= m \\ 10 &= m \end{aligned}$$

Since mass is measured in kg, we say that the mass of the box is 10 kg.

If your friend starts pushing at 40 N [left] your force will once again be balanced and you would plug 0 in for F: there is no unbalanced force. Whenever there is no unbalanced force, the acceleration is also 0. Whenever there *is* an unbalanced force, the acceleration is in the same direction. In this example, the unbalanced force was to the right so acceleration was also to the right.

Newton's Third Law

"For every action, there is an equal and opposite reaction."

Any force applied by object A onto object B results in a force of *equal size* begin applied by object B onto object A in the opposite direction. In other words, if you push the wall with a force of 40 Newtons [right], the wall pushes back on you with a force of 40 Newtons [left]!

The Four Fundamental Forces

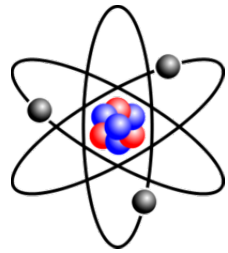
The forces below are the basis of all interaction between objects in the universe. There are many more forces (like the applied force, friction, and the normal force) that we see in everyday life that are combinations or other forms of these four forces. They are ordered below from the weakest force first to the strongest force last.

1. Gravitation

Everything that has mass (stuff that takes up space) exerts a gravitational force. You are gravitationally attracted to the Earth because the Earth has mass. You are also gravitationally attracted to the Moon, the Sun, your desk, your dog, and everything else in the universe that has mass. However, gravity is the weakest force for a reason. You can only see the effects of gravity when you are very close to an object that has a lot of mass. Take the Earth for example. It has a mass of 6,000,000,000,000,000,000,000,000 kg whereas the average 12 year old has a mass of 42 kg. The Earth has more "stuff" in it than you do. Therefore, gravity pulls you very strongly toward the Earth. On the other hand, say you have a mass of 42 kg and your best friend has a mass of 44 kg. You will both still be gravitationally attracted to each other because both of you have mass. But your friend does not have a lot of mass and therefore you don't feel yourself being pulled toward them. Once again, this is why gravity is the weakest force: it is only a big deal when you are very close to an object that has lots and lots of mass. Gravity is the force that caused my pencil to roll off my desk earlier. It is also the reason the Moon orbits the Earth and the Earth orbits the Sun. As I get further away from the Earth, its gravity has less of an effect on me.

2. Weak Nuclear Force

Matter is the stuff that makes up mass. Tiny **atoms** are the building blocks of matter; all atoms have a **nucleus** and lots of even smaller particles called **electrons** that zip around the nucleus. You may have seen atoms represented like this:



You may also have heard a little bit about different **elements** like Hydrogen, Helium, Oxygen, and Carbon. Different elements are different types of atoms. The **periodic table** has a full list of all elements in the universe:

Group→ ↓Period	1	2		3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	1 H																		2 He
2	3 Li	4 Be												5 B	6 C	7 N	8 O	9 F	10 Ne
3	11 Na	12 Mg												13 Al	14 Si	15 P	16 S	17 Cl	18 Ar
4	19 K	20 Ca		21 Sc	22 Ti	23 V	24 Cr	25 Mn	26 Fe	27 Co	28 Ni	29 Cu	30 Zn	31 Ga	32 Ge	33 As	34 Se	35 Br	36 Kr
5	37 Rb	38 Sr		39 Y	40 Zr	41 Nb	42 Mo	43 Tc	44 Ru	45 Rh	46 Pd	47 Ag	48 Cd	49 In	50 Sn	51 Sb	52 Te	53 I	54 Xe
6	55 Cs	56 Ba	*	71 Lu	72 Hf	73 Ta	74 W	75 Re	76 Os	77 Ir	78 Pt	79 Au	80 Hg	81 Tl	82 Pb	83 Bi	84 Po	85 At	86 Rn
7	87 Fr	88 Ra	**	103 Lr	104 Rf	105 Db	106 Sg	107 Bh	108 Hs	109 Mt	110 Ds	111 Rg	112 Cn	113 Uut	114 Fl	115 Uup	116 Lv	117 Uus	118 Uuo
			*	57 La	58 Ce	59 Pr	60 Nd	61 Pm	62 Sm	63 Eu	64 Gd	65 Tb	66 Dy	67 Ho	68 Er	69 Tm	70 Yb		
			**	89 Ac	90 Th	91 Pa	92 U	93 Np	94 Pu	95 Am	96 Cm	97 Bk	98 Cf	99 Es	100 Fm	101 Md	102 No		

The little 8 on top of the symbol for oxygen, O, means that an oxygen atom has 8 **protons** in it (the red circles inside the nucleus). Carbon (C) has 6 protons and Nitrogen (N) has 7 protons in it. The blue circles inside the nucleus are called **neutrons**. The weak nuclear force can change one type of atom into another. For example, it can turn a neutron in a carbon atom into a proton. Now this atom has 7 protons so it's not a carbon atom anymore – now it's a nitrogen atom. We say that the carbon atom has **decayed** into a nitrogen atom. Nuclear forces are very important in quantum physics, but it's not like we can't observe the weak force in our everyday lives. In order for the Sun to shine (give off light), it needs to turn one atom into another type of atom through decay. Without the weak force, our Sun wouldn't shine.

3. Electromagnetism

Without electromagnetism, there would be no light, no radiowaves, no microwaves, and no x-rays – these are all types of **electromagnetic radiation**. Also without it, there would be no electricity, and there would be no matter. You wouldn't exist!

When playing with magnets, you may have noticed that you can attract *and* repel other magnets: https://www.youtube.com/watch?v=LyvfDzRLsiU. This is due to the electromagnetic force. There are two types of **electric charges**: **positive** and **negative**. Positive charges are attracted to negative charges and repelled from other positive charges. Negative charges are attracted to positive charges and repelled by other negative charges. "Like repels like", we say. In an atom, the electrons are negatively charged and the protons (the red circles inside the nucleus) are positively charged, so they are attracted to each other. This is one of the reasons atoms stay together. Neutrons have no electric charge. In everyday life, you can see some types of electromagnetic radiation (visible light = colours) and you can feel the effects of electromagnetism when you touch other objects. There is a layer of electrons all around objects made of mass, including a layer around your hand and a layer around your pencil. These electrons are repelled by each other, so even though it feels like you are touching your pencil, you are actually not! You can never make contact with another physical object because your electrons will repel that object's electrons. So why can you feel the pencil in your hand? You can't. What you are feeling is the electromagnetic force as the electrons repel each other. Electromagnetism is the force that prevents my stapler from falling through my desk – the electrons in the stapler are repelled by the electrons in my desk (just like magnets repelling each other).

4. Strong Nuclear Force

You may have noticed that all those protons inside the tiny, tiny nucleus are positively charged and packed tightly together. How can this be? If like repels like, wouldn't the protons be repelled by other protons? Why are they happy to pack together so tightly like that? The reason is the strong force. Simply put, the strong force *forces* the protons inside to stay close to each other. It reminds me of when I'd get in an argument with my sister and my mom would make us sit next to each other and talk it out even though we didn't want to. It is the electromagnetic force that says protons should be repelled from each other, but since the strong force is much stronger than the electromagnetic force, protons stay together inside the nucleus.

Algebraic Activities

1. Consider the following diagram:

If ★ + ◯ − 2 = 7, what are the values of ★ and ◯?

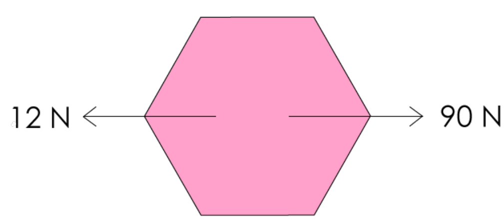
2. A large box of chocolates and a small box of chocolates together costs \$15. If the large box costs twice as much as the small box, what are the individual prices of the two boxes?

* 3. You want to rent a kayak and a paddle, but you only have \$50. There is a fixed fee to use the paddle, plus a charge of \$5/hr to use the kayak. For a three hour rental, the total price is \$30. What is the total cost for a six hour rental? Can you afford it?

Mechanics Activities

1. Olympic gold medalist Michael Johnson can run a 400m lap in 38 seconds.
   (a) What is his average speed?
   (b) How far could he run in a minute?
   (c) How far could he run in 10 seconds?
   (d) How long would it take him to run 21.1km?

2. Consider the following "Free Body Diagram":



   (a) Find the unbalanced force (also known as the "net force").
   (b) If in the diagram above, the hexagon had a mass of 5 kg, how much would it be accelerating and in what direction would it be accelerating?
   (c) With how much force would you have to push to make the hexagonal object accelerate at 10 m/s^2?

(d) If I had a new shiny blue hexagon that accelerated at 10 m/s^2 when the unbalanced force in part (a) was applied to it, how much mass would my shiny new hexagon have?

3. If Box A has twice the mass of Box B and you push both boxes with the same amount of force, which box will have the greater acceleration?

4. We said that having weight is a result of gravity acting on mass. We often hear the phrase "the Earth is weighing me down", which is literally true because weight is just the force of gravity. When you feel yourself being pulled toward the Earth by gravity, you are feeling your weight. We can measure any object's weight by finding the force of gravity that acts on them via Newton's second law $F = ma$. On Earth, we say that you are accelerating toward the ground as a result of gravity (you are being pulled toward the ground). Acceleration due to gravity on Earth is 9.8 m/s^2 [down]. So an object with mass 50 kg would weigh $F = (50) \times (9.8) = 490$ N on Earth. Notice that because weight is the *force* of gravity acting on an object, it is measured in Newtons (N).

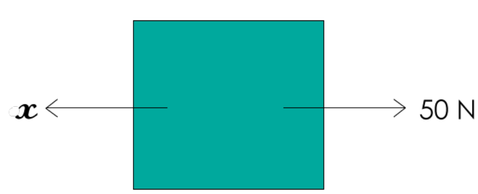
   (a) How much would an object of mass 42 kg weigh on Earth?

   (b) The average African bush elephant weighs 53,900 N on Earth. How much mass does it have?

   (c) The acceleration due to gravity on the Moon is one sixth of what it is on Earth. How much would an average 12 year old (42 kg) and an average African bush elephant weigh on the moon?

   Interesting note: On a scale, you see pounds (lbs) and kilograms (kg). Pounds (used in America) are a unit of weight (like Newtons are) but kilograms (used in Canada) are a unit of mass! Tsk, tsk...

* 5. Solve for x in Newtons if the size of acceleration of the block is 10 m/s^2:



What is the new x (call it x') if I add another block of equal mass on top? (Assume the same acceleration and forward force).
Hint: both your answers should have an m in them.

What is x if $m = 5$ kg? What is x'?

* denotes a hard question.