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Grade 6 Math Circles
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The History of Math: Gauss

Carl Friedrich Gauss

Painting by Christian Albrecht Jensen

Carl Friedrich Gauss was a German mathematician. Gauss has many mathematical achievements but today we will explore only two types of problems for which he became famous. One achievement he discovered while in school as a young boy. His teacher decides to give the class a task to busy them for a few minutes. The task is simple enough that it can be described in one sentence: add the numbers 1 to 100 in your head.

Go ahead, try and add the numbers 1 to 100. As legend has it, Gauss thought for a moment and while his peers were still very busy , he had his solution ready. Everyone is astonished, including his teacher. Gauss claims that the numbers 1 to 100, when summed, make a total of 5050. Is he correct? How did he do it?

Arithmetic Sequences

An *arithmetic sequence* is a list of numbers generated by adding a number to the previous term to get the next term. For example, the sequence $1, 2, 3, 4, 5, 6, 7, \ldots$ is arithmetic because I can add 1 to the previous number to get the next.

We can also think of an arithmetic sequence has having a *constant difference* between each of its numbers. For example, in the arithmetic sequence $2, 4, 6, 8, 10, \ldots$ we can see that $4-2=2$, $6-4=2$, $8-6=2$ and so on. The difference is always 2 so we say it is constant.

Exercise: Determine what the constant difference is in each of the following sequences and then give the next three numbers in the pattern:

Sequence	Difference	Next Three
$3, 6, 9, 12, \ldots$	3	15, 18, 21
$3, 5, 7, 9, \ldots$	2	11, 13, 15
$3, 15, 27, 39, \ldots$	12	51, 63, 75
$3, 4.1, 5.2, 6.3, \ldots$	1.1	7.4, 8.5, 9.6
* $5, 4, 3, 2, \ldots$	-1	1, 0, -1

Finding Any Term in a Sequence

A *formula* is like a mathematical algorithm; it has a specific purpose, takes input, and gives output. Arithmetic sequences have a strict pattern of adding. We will use this fact to invent a formula for calculating any term in any arithmetic sequence. For example, if I wanted to find the 27th number in this sequence, $1, 10, 19, 28, \ldots$, then I will not need to write down 27 numbers to figure it out because I can use a formula instead.

If I am given a sequence that counts by twos, $2, 4, 6, 8, \ldots$, then our common difference is 2. This is not a difficult sequence so it will be a good example for us to use to explain more difficult concepts. Say I wanted to find the 27th term; well, some of us may already know it is 54 because each number is its place in the sequence multiplied by 2. However, this trick may not always be easy to see in more difficult arithmetic sequences.

Think about how all arithmetic sequences are generated. We start with a number and add the common difference repeatedly to get new terms. Maybe our formula should reflect that. I want to be able to find any term so I will use the letter n to mean any number. I will use

n to mean the location of the number I want. The first term has a location of 1, the second term a location of 2, and so on. To find the n-th term in the sequence, we should start with the first term and repeatedly add the common difference enough times to get to the n-th term. What is a quick way of repeatedly adding? Multiplying.

If we want to find the second term, in our example we start with 2, how many common differences must we add to get the next term? We only add the common difference of 2 once. If we want the third term, we must add 2 two times to our first term. The fourth term will require adding 2 three times. See the pattern yet? If we start with our first term and want to find the n-th term then we will want to add the common difference n minus 1 times to the first term. So, to get the 27th term, we will want to add 2 twenty-six times $(27 - 1 = 26)$ since n is 27 here. This will get us the result we predicted: $2+(27-1)\times 2 = 2+26\times 2 = 2+52 = 54$.

If we write our terms as a_1 to mean the first term, a_n to mean the n-th term, where n is the location of the term we want, and d to mean the common difference then the formula looks something like this:

$$a_n = a_1 + (n-1) \times d, \text{ or using words:}$$

(term we want) = (first term) + (location of the term we want -1) $\times$ (common difference)

Try it out on these sequences by finding the 5th term by hand and then also use the formula.

Sequence	Difference	Fifth Term (by hand)	Fifth Term (by formula)
$3, 6, 9, \ldots$	3	12, 15	$3 + 4 \times 3 = 15$
$3, 5, 7, \ldots$	2	9, 11	$3 + 4 \times 2 = 11$
$3, 15, 27, \ldots$	12	39, 51	$3 + 4 \times 12 = 51$

Finding a Sum from a Sequence

Now we want to find a sum of the first few numbers in a sequence. Think about this small example: $1, 2, 3, 4$. We know that $1 + 2 + 3 + 4$ is 10 but let us try and develop another way of counting it. We should notice that $1+2+3+4 = 10 = 4+3+2+1$, or in other words I can write the sum backwards and get the same answer. It does not matter in which order I add numbers. Now, take both the forwards and backwards sums and pair up the numbers:

$$1+2+3+4$$
$$4+3+2+1$$

The pairs are above and below each other. Notice that when we add each of them, they all add up to 5! That is, $1+4 = 5$, $2+3 = 5$, $3+2 = 5$, and $4+1 = 5$. This technique works for all arithmetic sequences. We are always able to write the sum backwards and pair off numbers that all add up to the same number. Have a look at the same example when I add ALL of the numbers together in both the backwards and forwards sums:

$$1+2+3+4+4+3+2+1 = 5+5+5+5 = 10+10 = 20$$

We can see here that if I add two of the sums together, one forwards and one backwards, that their total sum is twice the total sum of one sequence. Well, maybe that is obvious to you. If we add two sums that are both equal to 10 together, of course it is the same as adding 10 + 10, or performing 2×10. But noticing this obvious pattern has actually given us a way to sum any arithmetic sequence!

Suppose we have a longer sequence. We know there are 20 terms in it, the first term is 2 and the 20th term is 40. Now we want to find its sum. Well we can pair up the numbers again:

$$2+\ldots+40$$
$$40+\ldots+2$$

We know all the numbers between 2 and 40 will also sum to 42 since the common difference in an arithmetic series is always the same between adjacent numbers in the sequence. So we know we have 20 terms in from each sequence that pair to sum to 42. Therefore, to find the sum of both our sequences combined (in other words, this sum: $2 + \ldots + 40 + 40 + \ldots + 2 = 42 + \ldots + 42$), we can use multiplication. Since there are going to be twenty 42's being added together, we can just calculate 20×42. This gives us 840. But remember, when we add two of our sums together we get two times the actual sum that we want.

Therefore, as our final step, we need to take our total sum and divide it by 2. So the sum of $2 + \ldots + 40$ will be $840 \div 2 = 420$.

We only needed the first and last numbers from our sequence and the number of terms in our sequence to calculate its sum. Here is a formula. a_1 and a_n, as before, stand for the first and last term of an arithmetic sequence and n is the number of terms in our sequence. We want to determine what the number is when each number is paired up and added — it will always be $a_1 + a_n$ since the first and last term always pair together. We then multiplied by n to calculate our sum times two. To get rid of the times 2, we can finally divide by 2. Let S be our sum, then:

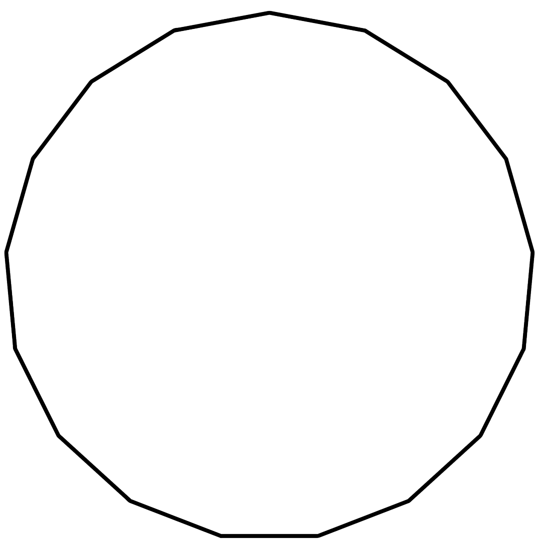
$$S = \frac{n \times (a_1 + a_n)}{2}$$

Exercise: Make sure the formula I found works by completing this table. By hand means adding each number one-by-one in your head or on a calculator. By formula means using just the first and last number and n to calculate the sum of the sequence.

Sequence Sum	n	Sum (by hand)	Sum (by formula)
$3 + 6 + 9 + 12 + \ldots + 21$	7		$7 \times (3 + 21) \div 2 = 84$
$3 + 5 + 7 + \ldots + 21$	10		$10 \times (3 + 21) \div 2 = 120$
$3 + 15 + 27 \ldots + 51$	5		$5 \times (3 + 51) \div 2 = 135$
$1 + 9 + 17 + \ldots + 41$	6		$6 \times (1 + 41) \div 2 = 126$

Constructing Regular Polygons

Gauss showed, when he was only 19, that it was possible to draw a heptadecagon with only a compass and straight-edge (a ruler without measurements). This technique of constructing shapes and other measurements with a compass and straight-edge is some of the earliest mathematics that existed. Numbers did not always exist as you and I understand them — as symbols that could mean many different things. Numbers were always referencing something of physical meaning. This could be distance, time, or volume. A Greek mathematician by the name of Euclid wrote an entire textbook for beginners on how to construct various quantities. The word construct, in this context, means to draw with only compass and straight-edge. The Greeks needed to use this method because they had no concept of algebra or equations. This book remained the standard for mathematics texts for over 1000 years!



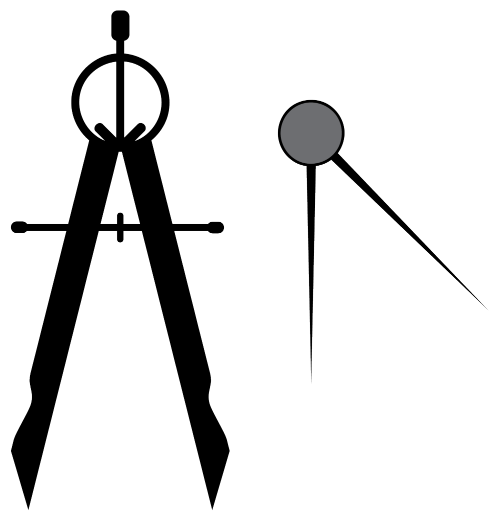


Figure 1: A 17-sided heptadecagon, the shape gauss constructed using only a compass and straight-edge. He showed it was possible to construct it before actually drawing one.

Nowadays, constructing numbers and shapes is more of a hobby than an actual useful tool. However, it can still be useful to know. Today we will learn to draw an equilateral triangle, a square, a regular pentagon, and a regular hexagon using only a compass and straight edge.

Before we begin, we should review a few key words and what they mean.

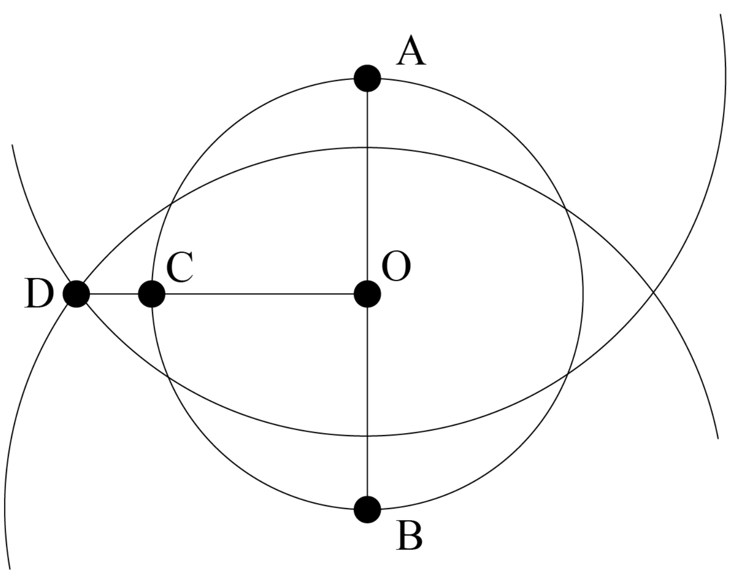
A *point* is simply a dot on a page; we usually name them with capital letters.

A *line segment* is part of a line drawn between two points; if a line's *endpoints* are A and B then we name the line segment AB. **Lines are always straight**, curves are curved.

An *intersection* is the point at which two lines or curves cross each other.

A *circle* is a type of curve that has the same radius throughout. A *radius* is a measurement from the centre of a shape to its edge. A *semi-circle* is half a circle; it is a type of arc. An *arc* is only a small piece of a circle. A *diameter* is a line drawn from one edge of the circle to the other edge passing through the centre; its length is twice the length of the radius.

Two lines are said to be *perpendicular* if they are at a 90 degree angle with each other. A *bisector* is a line that cuts another line segment in half. A *perpendicular bisector* is a bisector at a 90 degree angle from the line segment it is cutting in half.



O is a point and centre of the circle.

AB is a line segment from *A* to *B*.

D is a point of intersection.

The large circle is drawn about its centre *O*.

The circle has radius *CO*; *AO* and *OB* are also the radius.

AB is a diameter.

Two arcs intersect at point *D*.

DO is a perpendicular bisector to the line segment *AB*.

Constructing an Equilateral Triangle

1. Pick a point on the paper (leave some space all around it) and label it as O.

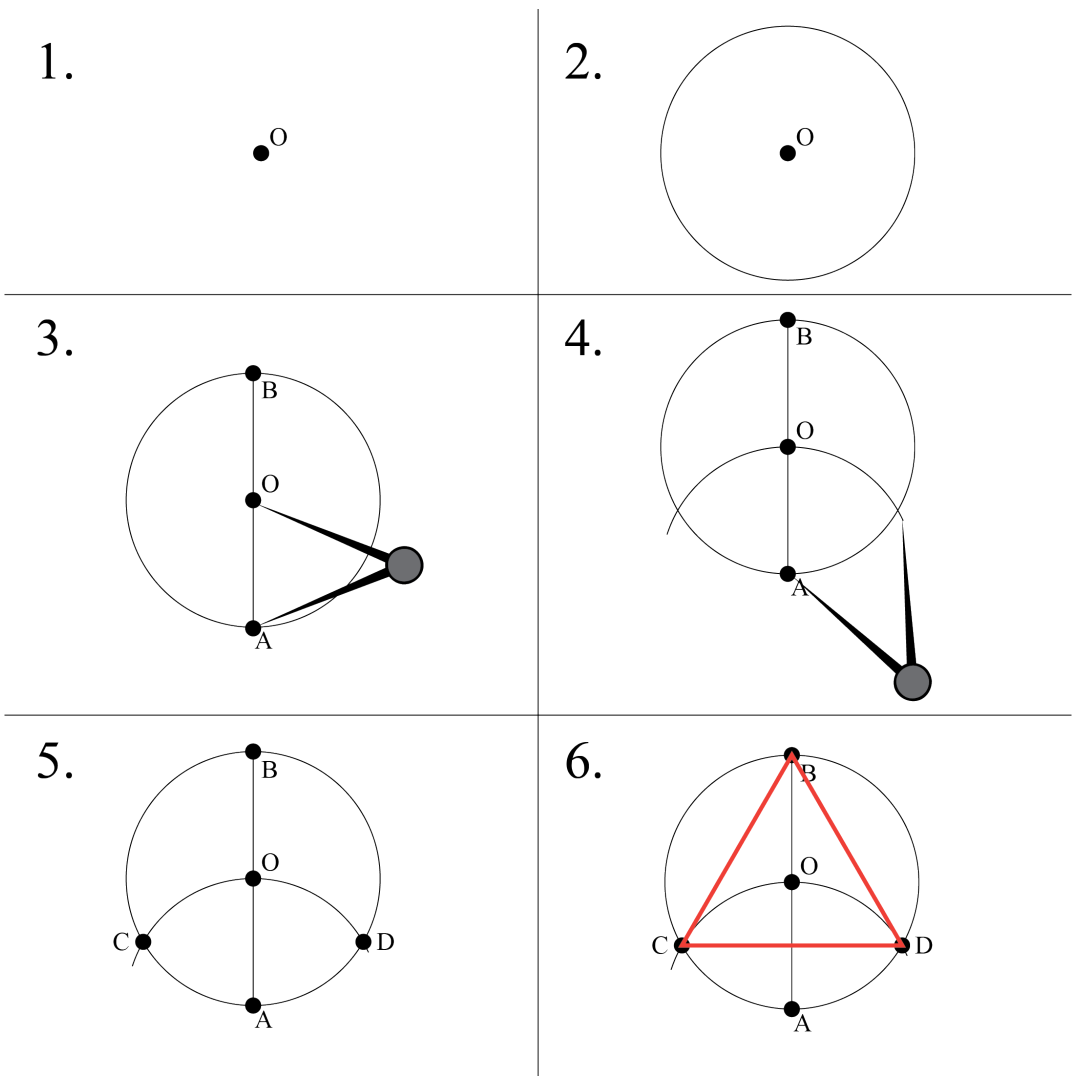
2. Use the compass to draw a circle about point O.

3. Draw a diameter of the circle and label each end as points A and B

4. Fix the compass so it is the length of the radius. Draw an arc about point A.

5. Label the intersections of the arc as C and D.

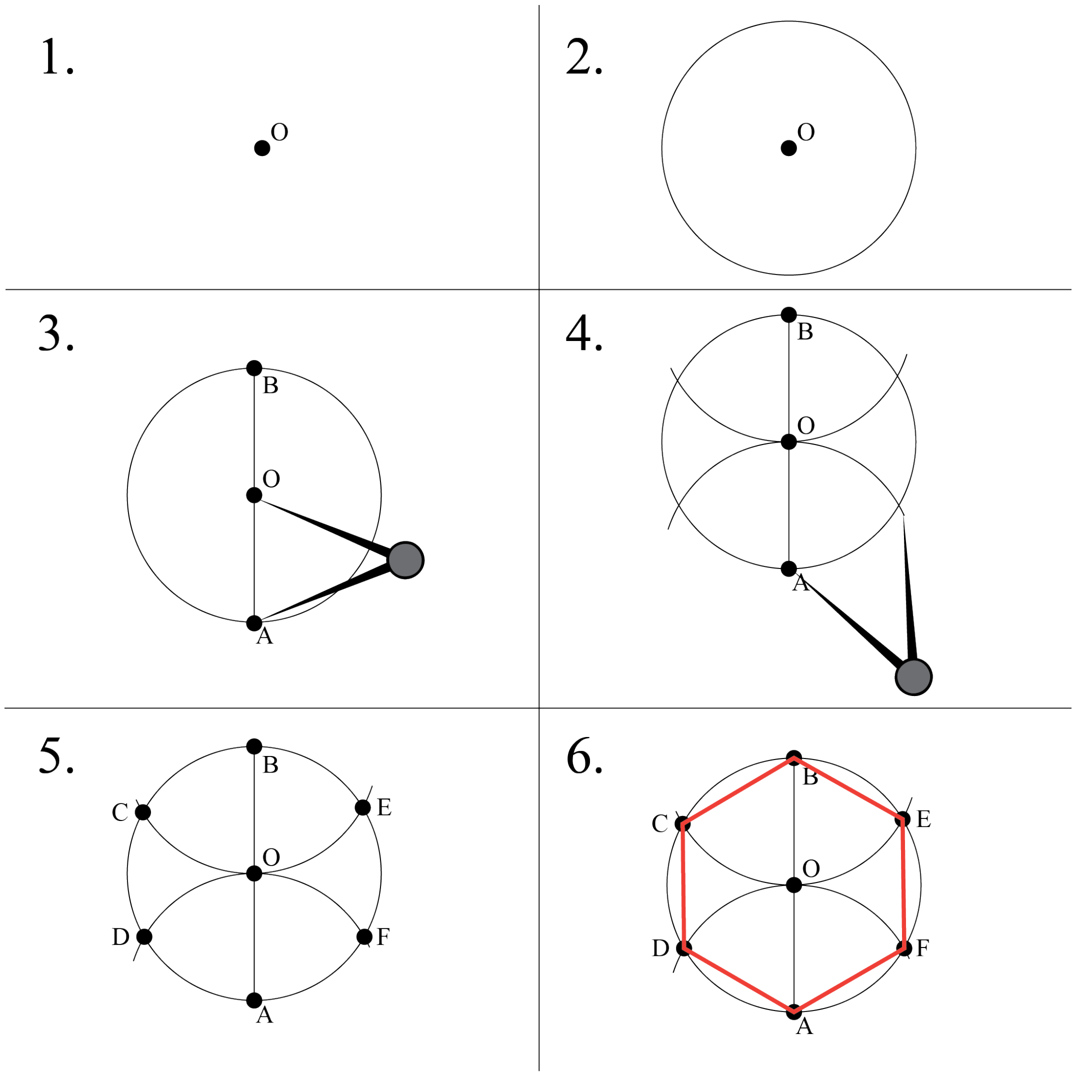
6. Using your straight-edge, connect the points B, C, and D.



Exercise: Construct a Regular Hexagon

Constructing a regular hexagon is like constructing the points for two equilateral triangles, one rightside up and one upside down, then connecting neighbouring points on the circle. Try on your own; see the solutions for a construction diagram.

Solution:



Constructing a Square

1. Pick a point on the paper (leave some space all around it) and label it as O.

1. Use the compass to draw a circle about point O.

1. Draw a diameter of the circle and label each end as points A and B.

2 - 4. Construct the perpendicular bisector of the diameter AB.

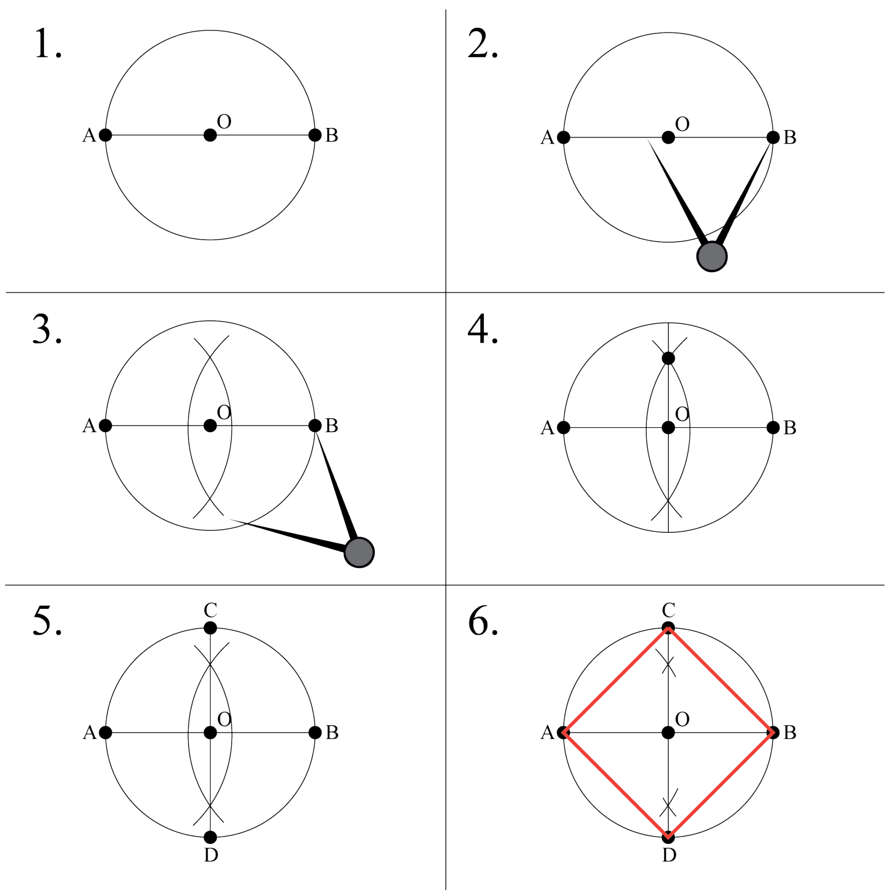
    2. Set the compass to a length longer than the half AB.

    3. Draw two arcs, one about A and the other about B.

    4. Connect the midpoint of AB to the intersections of the arcs.

5. Label the bisector's intersections with the circle as the points C and D.

6. Connect the points A, B, C, and D.



***Constructing a Pentagon**

None of the individual steps for constructing a pentagon is any more difficult than the previous polygons but there are more of these steps. Here is an algorithm that will produce a pentagon.

1. Draw a circle about point O and a vertical diameter with endpoints A and B.

2 - 3. Draw the perpendicular bisector of AB.

3. Label the bisector's intersection with the circle as C.

4 - 5. Draw the perpendicular bisector of CO.

6. Draw a circle of radius CD about D.

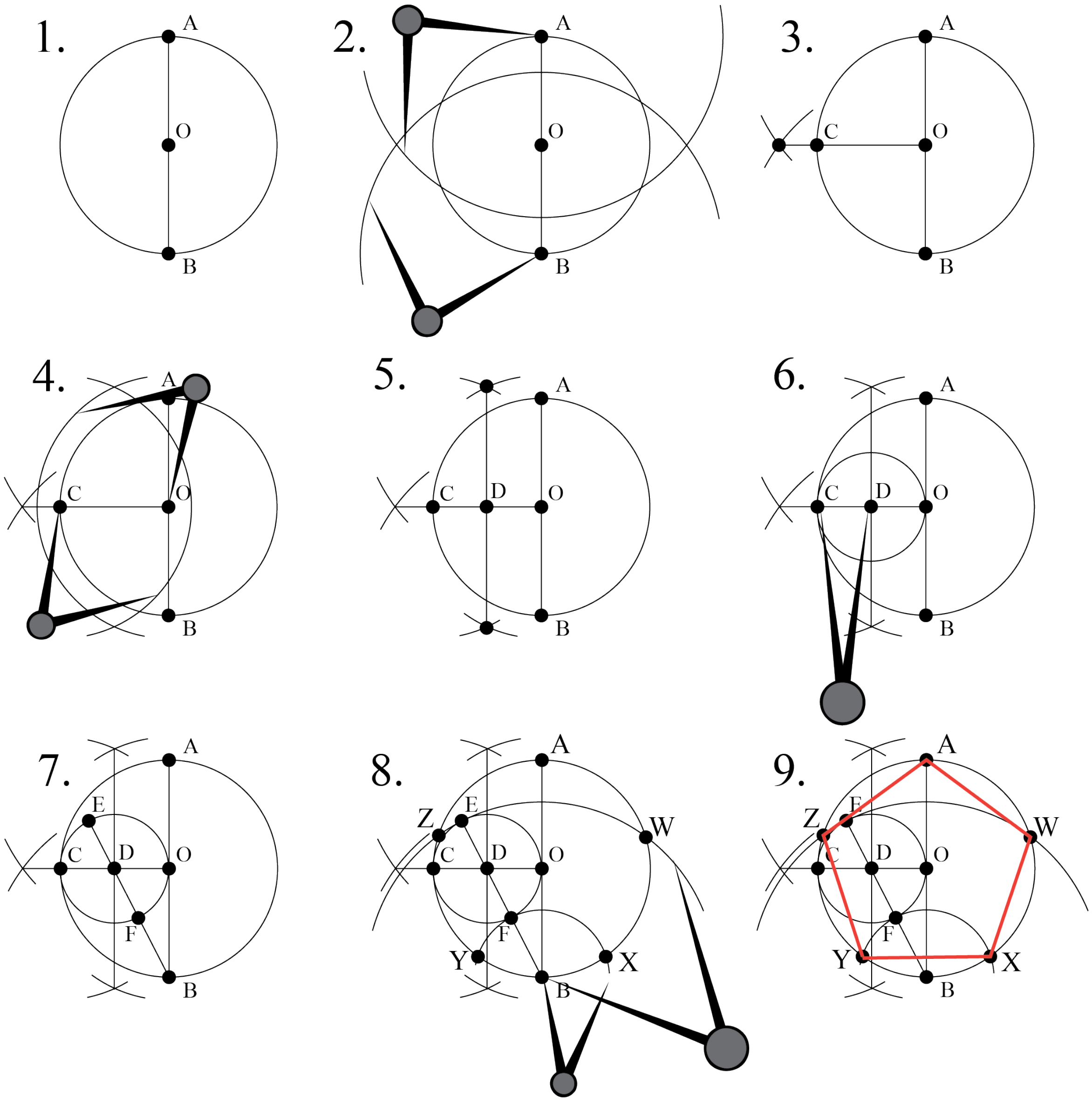
7. Draw the line through points D and B, extend the line to so it intersects the previous circle at E and F.

8. Draw two arcs:

   (a) One about B of radius FB. Label its intersections with the main circle Y and X.

   (b) The other also about B of radius EB. Label its intersections with the main circle Z and W.

9. Connect the points A to W to X to Y to Z back to A in that order.


1.
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Exercises

1. Determine whether or not each of the following sequences are arithmetic. Explain.

$2, 3, 5, 6, 8, 9, 11, 12$ Non-arithmetic, no common difference.	$652, 653, 654$ Arithmetic, common difference of 1.
$1, 8, 15, 22, 29, 36$ Arithmetic, $d = 7$.	$1, 1, 2, 3, 5, 8, 13$ Non-arithmetic, no common difference.
$2, 4, 8, 16, 32, 64, 128$ Non-arithmetic, no common difference.	$100, 75, 50, 25, 0$ Arithmetic, $d = -25$.
$341, 322, 303, 274$ Non-arithmetic, no common difference.	$\frac{1}{3}, \frac{5}{12}, \frac{1}{2}, \frac{7}{12}$ Arithmetic, $d = \frac{1}{12}$.

2. It is also possible to find any term in any arithmetic sequence starting with a term other than the first. For example, we could start with the 100th term and find the 50th term. Or we could start with the 7th term and find the 17th term. All you need to do is figure out how many common differences to add or subtract. We can do this by finding the differences in locations of the terms we want. The difference in locations of the terms is exactly how many common differences we will want to add or subtract.

   I will give you the common difference and a number in the sequence with its location. I will ask you to calculate the number in a specific location.

   (a) The common difference is $d = 3$. The 8th term in the sequence is 10. What is the 13th term in the sequence? Add 5 common differences. Therefore the 13th term is $8 + 5 \times 3 = 8 + 15 = 23$

   (b) $d = 5$. The $9th$ term in the sequence is 19. What is the 19th term in the sequence? Add 10 common differences. The 19th term is 59.

   (c) $d = 7$. The 3rd term in the sequence is 20. What is the 20th term in the sequence? Add 17 common differences. The 20th term is 122.

   (d) $d = 10$. The 31st term in the sequence is 320. What is the 1st term in the sequence? Subtract 30 common differences. The 1st term is 20.

3. Describe an algorithm for generating the following sequences, use the common difference of each sequence in your algorithm:

   (a) $1, 2, 3, 4, 5, 6, \ldots$ (c) $300, 301, 302, \ldots$ (e) $200, 201, 202, \ldots$

   (b) $0, 1, 2, 3, 4, 5, \ldots$ (d) $2, 4, 6, 8, 10, \ldots$ (f) $1, 3, 5, \ldots$

   For each sequence, use the same idea behind our formula for finding a*n*y term in the sequence ($a_n = a_1 + (n-1) \times d$). Pick the number to start with and then add generate each sequence by adding the common difference.

4. Find the sum of the following arithmetic sequences:

   (a) $1, 2, 3, 4, 5, 6, \ldots, 299,\ n = 299$ $S = 44850$ (c) $300, 301, 302, \ldots, 600,\ n = 301$ $S = 135450$

   (b) $0, 1, 2, 3, 4, 5, \ldots, 600,\ n = 601$ $S = 180300$ (d) $2, 4, 6, 8, 10, \ldots, 598,\ n = 299$ $S = 89700$

5. Construct an equilateral triangle using only a compass and a straight-edge. You may use the instructions and diagram provided.

6. Construct a square using only a compass and straight-edge.

7. * Construct a regular pentagon.

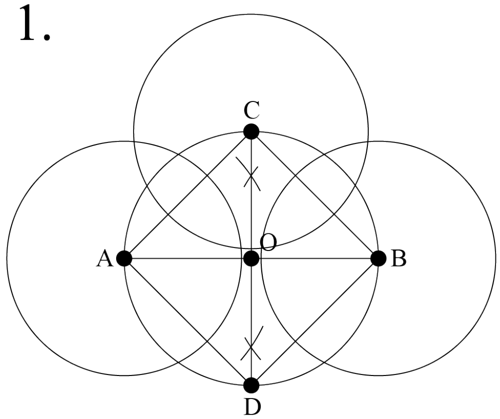
8. ** Construct a regular octagon.

   Hint 1: Constructing an octagon is similar to constructing a hexagon except we are putting two squares together. Review how we found the points for a square (it looks like a diamond) and then find a way to draw in another square (that has a flat bottom, not a diamond bottom) and connect neighbouring points.

   Hint 2: Construct the square and use perpendicular bisectors to find the last four points located on the circle.

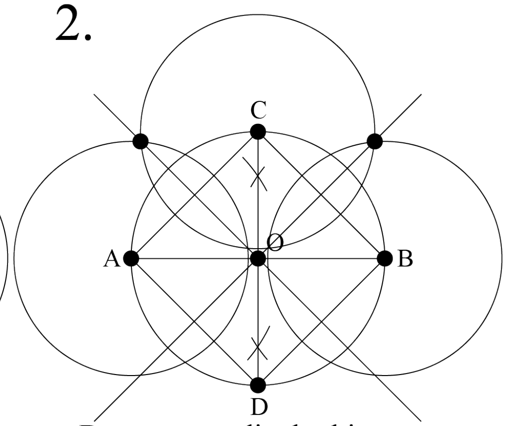
   Solution on next page.

1.



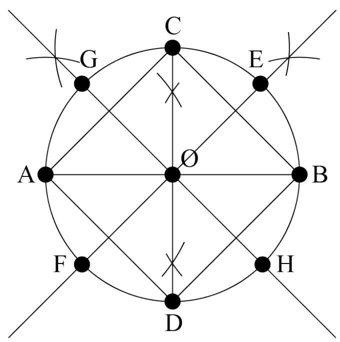
Draw circles of radius less than AO about the points A, C, and B.

2.



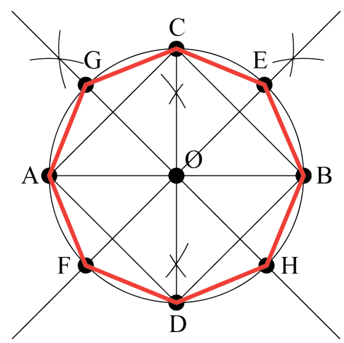
Draw perpendicular bisectors to the line segments AC and CB. Extend them outside the circle through point O.

3.



Find and label intersection points.

4.



Connect neighbouring points.

9. (Take this problem home!) Watch one or both of the following videos.

- *The Amazing Heptadecagon* by Numberphile (Brady Haran). `https://www.youtube.com/watch?v=87uo2TPrsl8`. This video explains Gauss's discovery that constructing the 17-gon is possible with only a compass and straight-edge.
- *Carl Friedrich Gauss Presentation* by Carissa B. `https://www.youtube.com/watch?v=16xZHmUj7Sc`. This video is an overview of Gauss's life and main achievements.