Math Circles
March 25th 2015
Paradoxes
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EXERCISES

Introduction Paradoxes:

1. **The Potato Paradox**:
   Suppose you have 100 pounds of potatoes, and they're special potatoes in that they are 99% water. Let's say you let them dehydrate for awhile, until they are only 98% water. How much do they weigh now? The surprising answer is: 50 pounds!

   (a) Is the statement true? If so, can you prove it? If not, why not?

   (b) What kind of paradox is this?

2. Let's think about a Disney cartoon for a second: Pinocchio! Pinocchio is a wooden doll whose nose will get longer if (and only if) he tells a lie.

   (a) What happens if Pinocchio says 'my nose will get longer now'?

   (b) What kind of paradox is this in part (a)?

3. Consider the following proof: We know that $\cos^2(x) + \sin^2(x) = 1$ for any $x \in \mathbb{R}$. So we get:

$$\begin{aligned} \cos^2(x) &= 1 - \sin^2(x) \\ \cos(x) &= \left(1 - \sin^2(x)\right)^{\frac{1}{2}} \\ 1 + \cos(x) &= 1 + \left(1 - \sin^2(x)\right)^{\frac{1}{2}} \end{aligned}$$

   Now let's substitute $x = \pi$ to get

$$1 - 1 = 1 + (1 - 0)^{\frac{1}{2}}$$

   or

$$0 = 2.$$

   (a) This can't be true, can it? What's wrong with the 'proof'?

   (b) What kind of paradox is this?

4. **All horses are the same colour**, did you know that? Let's prove it with mathematical induction!

   Let's let P_n be the statement 'every collection of n-many horses are all the same colour as each other'. Let's show that P_n is true for all n.

   Base Case: $n = 1$. Obviously every collection of a single horse are all the same colour as each other, so the $n = 1$ case is done.

   Inductive Step: Suppose that every collection containing exactly k-many horses are all the same colour as each other. We want to show that every collection of $k+1$ many horses all have the same colour.

   So take any collection of $k+1$-many horses. Now remove one horse at random. Then we're left with only k-many horses which, by assumption, all have the same colour as each other.

   Ok, put that horse we took away back in, and take a DIFFERENT horse out. Now, again, there are k-many horses so they are all the same color. So put that horse back, and it will be the same colour as the rest.

   So all $k+1$-many horses have the same colour, which finishes our mathematical induction proof.

   Questions:

   (a) Surely there's something wrong with the above argument, what is it?

   (b) What kind of paradox is this?

5. **Devil's Dice**: Consider three dice A, B, and C.

   - Die A has sides $2, 2, 4, 4, 9, 9$;
   - Die B has sides $1, 1, 6, 6, 8, 8$;
   - Die C has sides $3, 3, 5, 5, 7, 7$.

   (a) What is the probability that die A rolls higher than die B?

   (b) What is the probability that die B rolls higher than die C?

   (c) What is the probability that die C rolls higher than die A?

   (d) What kind of paradox is this?

6. **Hilbert's Grand Hotel**. Consider a hotel with an infinite number of rooms, labeled $1, 2, 3, \ldots$. Also suppose every room is occupied. You'd think that no new guests could fit, right? Well that's only true of our boring finite hotels.

   (a) How could you fit another guest into the hotel by shifting people to different rooms?

   (b) What about finitely many (say N) new guests?

   (c) What about *infinitely* many new guests?

   (d) What kind of paradox is this?